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On the Classification of Fundamental Particles.

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Summary. — A new four-dimensional isotopic spin formalism is developed for the heavy fundamental particles. All particles correspond to single-valued representations of the rotation group and there is complete symmetry between the isotopic properties of fermions and bosons. The resulting classification is similar to Gell-Mann's except that the degeneracy in the classification of the K-mesons is removed. The forms of the strong interactions are given explicitly. The physical consequences of the theory are discussed in detail. In particular it predicts the existence of a neutral K-meson of lifetime $\sim 10^{-19}$ s. A new experimental technique for detecting this particle is suggested.

1. – Introduction.

PAIS [1] has obtained a classification for the heavy fundamental particles by considering a four-dimensional isotopic spin space. One disadvantage of his scheme is that it predicts the existence of doubly charged hyperons and these have not been observed. In addition, if the reaction: two nucleons → → two hyperons is forbidden (and it has not been observed) there must

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(1) A. PAIS: *Proc. Nat. Acad. Sci.*, **40**, 484 (1954).

be two sets of heavy bosons (K-mesons) having the same quantum numbers (2).

A different scheme has been proposed by Gell-Mann (3). However if the experimental evidence that τ-mesons and θ-mesons have different spins and parities is accepted there is again a mysterious degeneracy in the classification of heavy bosons.

The purpose of this note is to develop a new four-dimensional isotopic spin formalism. In the Pais scheme fermions correspond to spinor representations of the rotation group (4) and bosons to tensor representations of the rotation group, in isotopic space. In our scheme all particles correspond to tensor representations and there is complete symmetry in the isotopic properties of fermions and bosons. No multiply charged particles appear and the resulting classification scheme is similar to Gell-Mann's, except that the τ-mesons and θ-mesons are assigned different isotopic spins and so the degeneracy is removed.

In section 2 some mathematical preliminaries concerning the representations of the rotation group in four dimensions are developed. In section 3 this formalism is applied to the problem of classifying the heavy fundamental particles. It is shown how their strong interactions can be written down. In section 4 the physical consequences of the theory for the K-mesons are developed. In particular the theory predicts the existence of a neutral K-meson of lifetime $\sim 10^{-19}$ s. Despite its instability, an experiment to show its existence is suggested.

2. – Isotopic Formalism.

The generators, $I_{\alpha\beta}$ ($\alpha, \beta = 1, ..., 4$), of the rotation group in four dimensions satisfy

$$I_{\alpha\beta} = -I_{\beta\alpha}\,, \tag{1}$$

and

$$[I_{\alpha\beta}, I_{\gamma\delta}] = i\delta_{\alpha\gamma}I_{\beta\delta} + i\delta_{\beta\delta}I_{\alpha\gamma} - i\delta_{\alpha\delta}I_{\beta\gamma} - i\delta_{\beta\gamma}I_{\alpha\delta}\,. \tag{2}$$

If we write

$$\tau_i = \tfrac{1}{2}(I_{4i} + I_{jk})\,, \tag{3}$$

$$\mu_i = \tfrac{1}{2}(I_{4i} - I_{jk})\,, \tag{4}$$

(2) M. GELL-MANN and A. PAIS: *Proceedings of the International Physics Conference, Glasgow*, 1954.

(3) M. GELL-MANN: *Phys. Rev.*, 92, 833 (1954) and reference (2).

(4) Because PAIS takes the charge to be $\tau_3 + \mu_3 + \frac{1}{2}$ (in our notation). This is to get conservation of baryons. However this is guaranteed by their fermion character.

where (i, j, k) is a cyclic permutation of (1, 2, 3), then

$$[\tau_i, \mu_l] = 0 , \tag{5}$$

and

$$[\tau_i, \tau_j] = i\tau_k , \tag{6}$$

$$[\mu_i, \mu_j] = i\mu_k . \tag{7}$$

Equations (6) and (7) are just the commutation relations of the generators of the rotation group in three dimensions. All irreducible representations of these operators may be labelled by a single quantum number, $\tau'(\mu')$, taking integral and half odd integral values. It is well known [5] that all irreducible representations of the rotation group in four dimensions can be labelled by the two quantum numbers τ' and μ'. Those representations for which $\tau' + \mu'$ is an integer are the single-valued tensor representations and those for which $\tau' + \mu'$ is half an odd integer are the double-valued spinor representations.

3. – Classification and Interactions.

We write our scheme in terms of the eigenvalues of the operators τ_3 and μ_3: (τ_3 plays the same part as isotopic spin, and μ_3 the same part as « strangeness », in the Gell-Mann scheme.) Electric charge is given by

$$Q = I_{43} = \tau_3 + \mu_3 , \tag{8}$$

for all particles. We classify fermions according to:

	τ_3	μ_3
N^+, N^0	$\frac{1}{2}$, $-\frac{1}{2}$	$\frac{1}{2}$
Ξ^0, Ξ^-	$\frac{1}{2}$, $-\frac{1}{2}$	$-\frac{1}{2}$
Σ^+, Σ^0, Σ^-	1, 0, -1	0
Λ^0	0	0

This is essentially the same as the Gell-Mann scheme. The differences appear in the bosons. We classify according to:

	τ_3	μ_3
π^+, π^0, π^-	1, 0, -1	0
τ^+, τ^0, τ^-	0	1, 0, -1
θ^+, θ^0	$\frac{1}{2}$, $-\frac{1}{2}$	$\frac{1}{2}$
$\bar{\theta}^0$, θ^-	$\frac{1}{2}$, $-\frac{1}{2}$	$-\frac{1}{2}$

[5] Cf. H. WEYL: *Theory of Classical Groups* (Princeton, 1939).

We postulate that the strong interactions are invariant under rotations in the four-dimensional isotopic space. Since I_{43} and I_{12} commute, they are separately conserved in these interactions and so τ_3 and μ_3 are separately conserved. The interactions with the electromagnetic field are not invariant under rotations but there is no inconsistency in supposing that they conserve τ_3 and μ_3 separately. The weak interactions, causing decays, conserve only $\tau_3+\mu_3$, the total charge.

The nucleons and cascade particles together correspond to the $(\frac{1}{2}, \frac{1}{2})$ representation of the rotation group. Therefore they can be represented by a vector N_α in isotopic space. In the equations of motion we shall have to write a term $a + b\mu_3$ for the mass to give the difference in mass between the nucleon and the cascade particle. The Σ particles correspond to the (1, 0) representation and so can be written as a self-dual antisymmetric tensor $\Sigma_{\alpha\beta}$, and the Λ^0 corresponds to the (0, 0) representation and is a scalar Λ.

Among the bosons the π-mesons are represented by a self-dual antisymmetric tensor $\pi_{\alpha\beta}$, and the θ-mesons are represented by a vector θ_α. The τ-mesons correspond to the (0, 1) representation of the rotation group and so are represented by an anti-self-dual antisymmetric tensor $\tau_{\alpha\beta}$.

In our scheme all particles correspond to single-valued representations of the rotation group. This is essential because only such particles will have integral electric charge. Nearly all representations corresponding to unit charge have experimentally observed counterparts. The only exceptions are a fermion family corresponding to the (0, 1) representation and a scalar boson (0, 0). If the fermion family had a mass greater than about 2 800 electron masses it would be highly unstable under θ-decay into a nucleon.

The problem of writing down the strong interactions is simply that of forming appropriate invariants from the vectors and tensors at our disposal. We give two examples, writing only the isotopic indices. The interaction of τ-mesons with nucleons and cascade particles is given by

$$\bar{N}_\alpha \tau_{\alpha\beta} N_\beta \,, \tag{9}$$

and the interaction of θ-mesons with nucleons, cascade particles and Λ-particles is given by

$$\bar{\Lambda}\theta_\alpha N_\alpha + \text{c.c.} \,. \tag{10}$$

4. – K-mesons.

The similarity of our scheme and Gell-Mann's means that many of the predictions are the same. Consequently we shall only discuss in detail the physical properties of the K-mesons since these show up the differences between the two schemes.

We take the τ^0 and its charge conjugate to be same particle since $N^0 \to \Lambda^0 + \tau^0$ is forbidden. However the τ^0 is a highly unstable particle with a fast γ-decay ($\sim 10^{-15}$ s) and a very fast 3π-decay ($\sim 10^{-19}$ s). The mode of production with the lowest threshold is

$$\pi + N \to N + \tau^0 . \tag{11}$$

This should take place above about 0.5 GeV. The existence of this process, together with the other K-meson processes discussed below that take place at higher thresholds, should explain the broad second resonance in high energy $\pi^- - N^+$ scattering ([6]). We note that the reaction (11) corresponds to $\tau' = \frac{1}{2}$. The τ^0 produced will rapidly decay into 3π-mesons and this process may explain the high probability of $N^0 \to \pi^+ + \pi^- + \pi^0$ events in π^--N^+ scattering at 1·4 GeV ([7]).

A possible way of verifying the existence of the τ^0 despite its short lifetime would be as follows: He atoms are bombarded with π^- with energies just at the threshold for the production of σ_0 by (11). Any τ^0 created will be slow and will certainly decay outside the range of nuclear forces but inside the mesic atomic radius ($\sim 10^{-10}$ cm). A slow π^- produced by this decay would be captured by the remaining proton to form a mesic atom. This way of forming the mesic atom should be much more probable at this energy than direct π^- capture. Consequently there should be a sharp increase in the amount of characteristic mesic atom X-ray emission at the threshold for τ^0 production.

The τ^+ may be produced by reactions such as

$$\pi + N \to \tau^+ + \Xi , \tag{12}$$

while the θ^+ is produced by reactions such as

$$\pi + N \to \theta^+ + \Lambda^0 , \tag{13}$$

$$\pi + N \to \theta^+ + \Sigma . \tag{14}$$

The question whether τ^+ is produced in association with a Ξ or with a Σ or Λ gives another discriminating test between our theory and Gell-Mann's. As yet the evidence ([8]) is too indefinite to decide. The thresholds for (13) and (14)

([6]) M. Gell-Mann and K. M. Watson: *Annual Review of Nuclear Science*. 4 (1954).

([7]) W. B. Fowler, R. M. Lea, W. D. Shepherd, R. P. Shutt, A. M. Thorndike and W. L. Whittemore: *Phys. Rev.*, **97**, 797 (1955).

([8]) W. B. Fowler, R. P. Shutt, A. M. Thorndike and W. L. Whittemore: *Phys. Rev.*, **98**, 121 (1955).

are lower than that for (12) and this agrees with the evidence that θ's are more common than τ's. Since there is some analogy in our scheme between τ's and π's we expect that the K-meson having a weak μ-decay is the τ in analogy with $\pi \to \mu + \nu$.

The τ^- and θ^- can only be produced in pairs with the τ^+ and θ^+ respectively, by such reactions as

$$\pi + N \to \tau^+ + \tau^- + N , \tag{15}$$

$$\pi + N \to \theta^+ + \theta^- + N . \tag{16}$$

This explains the relative abundance of positively charged K-mesons over negatively charged K-mesons, since the former can also be produced by (12), (13) and (14), with lower threshold.

* * *

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RIASSUNTO (*)

Si sviluppa per le particelle pesanti fondamentali un nuovo formalismo assumendo lo spin isotopico quadridimensionale. Tutte le particelle corrispondono a rappresentazioni ad un unico valore del gruppo rotazionale e si ha completa simmetria tra le proprietà isotopiche dei fermioni e dei bosoni. La classificazione risultante è simile a quella di Gell-Mann, salvo che scompare la degenerazione nella classificazione dei mesoni K. Le forme dell'interazione forte sono date esplicitamente. Si discutono dettagliatamente le conseguenze fisiche della teoria. In particolare questa predice l'esistenza di un mesone K neutro di vita media $\sim 10^{-19}$ s. Si suggerisce una nuova tecnica sperimentale per la scoperta di questa particella.

(*) *Traduzione a cura della Redazione.*

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Zur Berechnung des Grundzustandes und der Masse des Polarons.

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Zusammenfassung. — Der in einer früheren Arbeit untersuchte Variationsansatz zeigte zwar in den Grenzfällen schwacher und starker Kopplung das Verhalten der exakten Lösung, brachte bei schwacher Kopplung jedoch keine Verbesserung gegenüber dem Ansatz von Lee, Low und Pines. Durch Einführung eines anderen Typs von Korrelationen zwischen den Schallquanten haben wir unsern Ansatz jetzt soweit verbessert, daß der Grundzustand des Polarons etwa ebensogut beschrieben wird wie nach dem inzwischen veröffentlichten Verfahren von Feynman. Die bisherigen Resultate für die Energie des Grundzustandes und die Masse des Polarons werden diskutiert. An den Pekarschen Produktansatz wird ein Störungsverfahren für starke Kopplung angeschlossen.

1. – Einleitung.

Ein nichtrelativistisches Teilchen sei mit beliebiger Stärke an ein reelles, skalares Feld gekoppelt. Die tiefsten Zustände dieses Systems sollen mit einem Variationsverfahren untersucht werden. Obwohl unsere Methode allgemeiner anwendbar ist, spezialisieren wir auf den in einer früheren Arbeit (1) behandelten Hamiltonoperator des Polarons (2,3)

$$(1) \qquad H = \frac{\mathfrak{p}^2}{2m} + \hbar\omega \sum b_{\mathfrak{k}}^{+} b_{\mathfrak{k}} - i\hbar\omega \left(\frac{4\pi}{uV}\right)^{\frac{1}{2}} g \sum \frac{1}{k} \left(b_{\mathfrak{k}} \exp\left[i\mathfrak{k}\mathfrak{r}\right] - b_{\mathfrak{k}}^{+} \exp\left[-i\mathfrak{k}\mathfrak{r}\right]\right),$$

(1) G. Höhler: *Zeits. f. Phys.*, **140**, 192 (1955) als I zitiert. S.a. *Zeits. Naturfor.* **9a**, 801 (1954).

(2) H. Fröhlich: *Advances in Physics*, **3**, 325 (1954). Bericht über die Theorie des Polarons; W. Schottky: *Halbleiterprobleme*, II. Bericht von H. Haken (Brawnschweig, 1951).

(3) S. I. Pekar: *Untersuchungen über die Elektronentheorie der Kristalle*. Akademie-Verlag, Berlin, 1953. (Deutsche Übersetzung).

g: Kopplungsfaktor, V: Normierungsvolumen, $u = (2m\omega/\hbar)^{\frac{1}{2}}$, Summation über alle $\mathfrak{k}$.

Er ist mit dem Operator der Gesamtwellenzahl

$$\mathfrak{W} = \mathfrak{p}/\hbar + \sum \mathfrak{k} b_{\mathfrak{k}}^{+} b_{\mathfrak{k}} , \tag{2}$$

vertauschbar. Wir können daher die gesuchten Zustände so wählen, daß sie zugleich Eigenzustände von $\mathfrak{W}$ sind. Geben wir etwa einen Eigenwert $\mathfrak{W}'$ von $\mathfrak{W}$ vor, so besteht das Problem darin, in dem dadurch festgelegten Unterraum des Hilbertraumes den tiefsten Zustand von H zu suchen. Es ist zweckmäßig, zunächst eine unitäre Transformation ([5]) mit

$$S = \exp\{i(\mathfrak{W}' - \sum \mathfrak{k} b_{\mathfrak{k}}^{+} b_{\mathfrak{k}} x\} , \tag{3}$$

vorzunehmen. Bei gegebener c-Zahl $\mathfrak{W}'$ ist dann der Grundzustand von ([6])

$$\mathcal{H} = (\mathfrak{W}' - \sum \mathfrak{k} b_{\mathfrak{k}}^{+} b_{\mathfrak{k}})^2 + \sum b_{\mathfrak{k}}^{+} b_{\mathfrak{k}} - i\varkappa \sum \frac{1}{k} (b_{\mathfrak{k}} - b_{\mathfrak{k}}^{+}) , \tag{4}$$

gesucht. $\varkappa = (4\pi g^2/V)^{\frac{1}{2}}$. Wir beschränken uns vorläufig auf $\mathfrak{W}' = 0$.

In I haben wir den Variationsansatz (I, 10.3)

$$\Phi_V = \int G(\mathfrak{R}) \exp\left[\sum F(\mathfrak{k}, \mathfrak{R}) b_{\mathfrak{k}}^{+}\right] \mathrm{d}\mathfrak{R} \Phi_0 , \tag{5}$$

vorgeschlagen (Φ_0 = Vakuumzustand) und seine Beziehung zu den Ansätzen von Lee, Low, Pines ([5]), Gurari ([7]) und Pekar ([3,4]) untersucht. Eigentlich müßte man zu jeder Kopplungsstärke g die besten Funktionen F und G bestimmen. Statt dessen haben wir für F und G einfache Funktionen angesetzt und lediglich einige darin enthaltene Variationsparameter variiert. Es stellte sich heraus, daß unser Ansatz sicher nicht schlechter ist als der von LLP und *zugleich* stets besser als der Produktansatz von Pekar.

Wir entwickeln den Zustand (5) nach Zuständen mit 0, 1, 2, ... Teilchen

$$\Phi_V = \int G(\mathfrak{R}) \mathrm{d}\mathfrak{R} \Phi_0 + \sum_{\mathfrak{k}} \int G(\mathfrak{R}) F(\mathfrak{k}, \mathfrak{R}) \mathrm{d}\mathfrak{R} b_{\mathfrak{k}}^{+} \Phi_0 + \tag{6}$$

$$+ \frac{1}{2} \sum_{\mathfrak{k},\mathfrak{k}'} \int G(\mathfrak{R}) F(\mathfrak{k}, \mathfrak{R}) F(\mathfrak{k}', \mathfrak{R}) \mathrm{d}\mathfrak{R} b_{\mathfrak{k}}^{+} b_{\mathfrak{k}}^{+} \Phi_0 + \ldots .$$

([4]) S. I. Pekar: *Fortschr. Phys.*, **1**, 367 (1954). (Deutsche Übersetzung eines Berichts).

([5]) T. D. Lee, F. Low und D. Pines: *Phys. Rev.*, **90**, 297 (1953). Als LLP zitiert.

([6]) Wir gehen jetzt zu dimensionslosen Größen über. Vgl. I, S. 194.

([7]) M. Gurari: *Phil. Mag.*, **44**, 329 (1953).

Für den in I behandelten Spezialfall (I, 10.4)

$$(7) \qquad G(\mathfrak{R}) = \psi(R)\exp[i\mathfrak{W}'\mathfrak{R}]; \quad F(\mathfrak{k},\mathfrak{R}) = -i\varkappa\frac{\varrho(\mathfrak{k})}{k}\exp[-i\mathfrak{k}\mathfrak{R}],$$

ergibt (6) mit $\mathfrak{W}' = 0$ und $\int\psi(R)\,\mathrm{d}\mathfrak{R} = 1$

$$(8) \qquad \Phi_V = \Phi_0 - i\varkappa(2\pi)^3\sum_{\mathfrak{k}}\frac{\varrho(\mathfrak{k})}{k}\tilde{\psi}(\mathfrak{k})b_{\mathfrak{k}}^{+}\Phi_0 - \\ - \frac{\varkappa^2}{2}(2\pi)^3\sum_{\mathfrak{k},\mathfrak{k}'}\frac{\varrho(\mathfrak{k})}{k}\frac{\varrho(\mathfrak{k}')}{k'}\tilde{\psi}(\mathfrak{k}+\mathfrak{k}')b_{\mathfrak{k}}^{+}b_{\mathfrak{k}'}^{+}\Phi_0 \pm \ldots,$$

wobei

$$\tilde{\psi}(\mathfrak{k}) = \frac{1}{(2\pi)^3}\int\exp[-i\mathfrak{k}\mathfrak{R}]\;\psi(R)\,\mathrm{d}\mathfrak{R}.$$

Während bei LLP die Koeffizienten in den k's separiert sind (z.B. im 2-Teilchen-Term (I, 3.8) $\frac{1}{2}\sum f(k)f(k')b_{\mathfrak{k}}^{+}b_{\mathfrak{k}'}^{+}\Phi_0$, (« Hartree-Näherung »), enthält unser Ansatz (8) *Korrelationen* eines ganz bestimmten Typs. Ihre Stärke wird durch einen Variationsparameter in ϱ oder ψ geregelt. Bei starker Kopplung führt die Hinzunahme dieses Typs von Korrelationen zu einer erheblichen Absenkung der Energie (Von $E = -g^2$ auf $E = -0.1g^4$). Man könnte denken, daß bei schwacher Kopplung mit kleineren Korrelationsstärken wenigstens eine kleine Verbesserung gegenüber dem Ansatz ohne Korrelationen eintritt. Das ist jedoch *nicht* der Fall. Wir fanden in I bis zur Kopplung $g^2 \approx 5$ *keine* Verbesserung gegenüber LLP.

2. – Variationsansatz.

Um einen für schwache Kopplung geeigneten Typ von Korrelationen zu finden und durch F und G auszudrücken, gehen wir von dem Zustand 2. Näherung der üblichen Störungstheorie aus, die den Kopplungsterm als Störglied behandelt:

$$(9) \qquad \Phi_{St}^{(2)} = \left(1 - \frac{g^2}{4}\right)\Phi_0 + \sum f(k)b_{\mathfrak{k}}^{+}\Phi_0 + \\ + \tfrac{1}{2}\sum f(k)f(k')\left\{1 - \frac{2(\mathfrak{k},\mathfrak{k}')}{2 + (\mathfrak{k}+\mathfrak{k}')^2}\right\}b_{\mathfrak{k}}^{+}b_{\mathfrak{k}'}^{+}\Phi_0,$$

$$(10) \qquad f(k) = \frac{-i\varkappa}{k(1+k^2)} = -i\left(\frac{4\pi}{V}\right)^{\frac{1}{2}}\frac{g}{k(1+k^2)}.$$

Wir haben bisher keine Funktionen F und G gefunden (8), die mit (6) gerade (9) ergeben. Daher begnügen wir uns damit,

$$(11) \qquad F(\mathfrak{k}, \mathfrak{K}) = f(k)\left[1 + i(\mathfrak{k}, \mathfrak{K})\frac{\sqrt{3}}{R}\exp\left[-k^2R^2\right]\right]; \quad G(\mathfrak{K}) = \frac{\exp\left[-2R^2\right]}{\pi R}$$

zu setzen. Dann erhält das mit (6) gebildete

$$(12) \qquad \Phi_\nu = \Phi_0 + \sum f(k)b_{\mathfrak{k}}^+\Phi_0 + \tfrac{1}{2}\sum f(k)f(k')\left\{1 - \frac{2(\mathfrak{k}, \mathfrak{k}')}{2 + k^2 + k'^2}\right\}b_{\mathfrak{k}}^+b_{\mathfrak{k}'}^+\Phi_0 + \ldots$$

Korrelationen, die denen in (9) ähnlich sind. Der Faktor $(1 - g^2/4) = 1 - \frac{1}{2}\sum|f|^2$ vor Φ_0 in (9) sorgt für die Normierung bis zur Ordnung g^2. Wir können in (12) auf ihn verzichten, da wir den Zustand später normieren werden. Die Benutzung des Zustandes (12) wird nahegelegt durch eine Störungstheorie mit $\sum(\mathfrak{k}, \mathfrak{k}')b_{\mathfrak{k}}^+b_{\mathfrak{k}'}^+b_{\mathfrak{k}}b_{\mathfrak{k}'}$ als Störterm (I, 4.7). Die Entwicklung des gestörten Zustandes 1. Näherung beginnt dann genau wie (12) (9).

Wir wollen den Ansatz (5), (11) nicht vollständig durchrechnen sondern nur feststellen, wie die ersten Terme einer Entwicklung von E nach steigenden Potenzen von g lauten. Insbesondere interessiert uns der Koeffizient des g^4-Terms. Da die Reihe (12) nach steigenden Potenzen von g fortschreitet, genügen dazu die ersten drei Glieder. Zuerst geben wir etwas allgemeiner den Erwartungswert von

$$(13) \qquad \Phi = \Phi_0 + \sum f(k)b_{\mathfrak{k}}^+\Phi_0 + \tfrac{1}{2}\sum f(k)f(k')[1 + h(\mathfrak{k}, \mathfrak{k}')]b_{\mathfrak{k}}^+b_{\mathfrak{k}'}^+\Phi_0$$

mit $\mathcal{H}$ an. $h(\mathfrak{k}, \mathfrak{k}')$ sei reell und von g unabhängig, f ist durch (10) gegeben.

$$(14) \qquad E = \frac{(\Phi, \mathcal{H}\Phi)}{(\Phi, \Phi)} = -g^2 + g^4\frac{8\pi^2}{V^2}\sum\frac{4(\mathfrak{k}, \mathfrak{k}')h + [2 + (\mathfrak{k} + \mathfrak{k}')^2]h^2}{k^2(1 + k^2)^2k'^2(1 + k'^2)^2} + \ldots .$$

Die weiteren Terme enthalten höhere Potenzen als g^4. In (12) ist speziell

$$(15) \qquad h(\mathfrak{k}, \mathfrak{k}') = \frac{-2(\mathfrak{k}, \mathfrak{k}')}{2 + k^2 + k'^2} .$$

Die Integrationen können ausgeführt werden. Man erhält die ersten Terme

(8) Herr Dr. HAKEN (Erlangen) hat kürzlich auf der Arbeitstagung in Oberwolfach einen Beweis dafür mitgeteilt, daß sich jeder beliebige Variationsansatz in der Form (5) schreiben läßt.

(9) E. HAGA: *Porgr. Theor. Phys.*, **11**, 449 (1954). Die ersten beiden Terme von (16) stimmen mit Hagas Formel (24) für die Energie 2. Näherung überein. Herr Dr. HAGA

einer Entwicklung des Erwartungswertes von $\mathcal{H}$ mit dem Zustand (5), (11) ([9]):

$$E = -g^2 - \frac{2}{3}\left(\frac{1}{8} - \frac{1}{3\pi}\right) g^4 \pm \ldots = -g^2 - 0.0126 g^4 \pm \ldots . \tag{16}$$

Damit können wir die in der Schlußbemerkung von I aufgeworfene Frage beantworten. (7) ist bei schwacher Kopplung ungeeignet, es gibt aber durchaus Zustände der Form (5), die ebenso gute Ergebnisse liefern wie die besten anderen Variationsverfahren. Man gewinnt einen guten Ansatz für alle Kopplungsstärken, wenn man mit Hilfe eines Variationsparameters die Ansätze (I, 12a) und (11) kombiniert. Anstelle dieser etwas gewaltsamen Zusammensetzung sollte man einen relativ einfachen Zustand vom Typ (5) suchen, der einerseits bei einer Entwicklung etwa wie (9) beginnt und andererseits auch speziell in (7) übergehen kann.

Aus (14) kann man das bestmögliche $h(\mathfrak{k}, \mathfrak{k}')$ bestimmen:

$$h(\mathfrak{k}, \mathfrak{k}') = \frac{-2(\mathfrak{k}, \mathfrak{k}')}{2 + (\mathfrak{k} + \mathfrak{k}')^2} . \tag{17}$$

Es führt auf den störungstheoretischen Zustand (9) zurück (bis auf den Normierungsfaktor bei Φ_0, s.o.). Wir setzen (17) in (14) ein ([10]):

$$E = -g^2 - g^4 \frac{32\pi^2}{V^2} \sum \frac{(\mathfrak{k}, \mathfrak{k}')^2}{k^2(1 + k^2)^2 k'^2 (1 + k'^2)^2 [2 + (\mathfrak{k} + \mathfrak{k}')^2]} \pm \ldots . \tag{18}$$

Dieses Resultat stimmt bis zum g^4-Term mit der Energie überein, welche man durch Störungsrechnung 4. Näherung erhält (Kopplungsterm als Störung). Zunächst sollte man erwarten, daß man den gestörten Zustand 3. Näherung benötigt ([11]), aber man kann sich davon überzeugen, daß die in 3. Näherung gegenüber (9) hinzukommenden Glieder keinen g^4-Beitrag geben. A. Müllensiefen ([12]) hat die Integrale in (18) ausgewertet:

$$E = -g^2 - 0.0157 g^4 . \tag{19}$$

hat freundlicherweise bestätigt, daß der auf seinen Sonderdrucken handschriftlich zugefügte Faktor $\frac{1}{2}$ gestrichen werden muß.

([10]) Die Störungstheorie von LLP ergibt gerade die Hälfte dieses g^4-Termes.

Anm. bei der Korrektur. – Bei einer Nachprüfung der zu (42) LLP führenden Rechnung fanden wir im Nenner $2m^2$ statt $4m^2$, sodaß nunmehr für $\mathfrak{W}' = 0$ Übereinstimmung mit unserer Formel (18) besteht. Das ist befriedigend, weil die Störungstheorie von LLG in höherer Ordnung keinen g^4-Beitrag mehr liefert. Das von Fröhlich ([2]) (6.24) für die Gültigkeit der Störungsrechnung angegebene Kriterium lautet jetzt: $g^4 \ll 63$. Die Masse hat bei $g^2 = 6$ den Wert $3{,}1m$ statt $2{,}4m$ (ohne Storungsrechnung: $2m$)

([11]) D. Bohm: *Quantum Theory* (New York, 1951), p. 456.

([12]) *Diplomarbeit* (Göttingen, 1955).

3. – Diskussion.

In I haben wir (wie LLP) nach Beschränkung auf einen Eigenzustand von $\mathfrak{W}$ die Variablen des Teilchens eliminiert und dann einen Variationsansatz für den Grundzustand des Feldes gemacht. Feynman hat kürzlich gezeigt ([13]), wie man den umgekehrten Weg gehen kann. Er hat zunächst mit seinem in der Quantenelektrodynamik so erfolgreichen Verfahren die Variablen des Feldes « eliminiert » und dann für das Teilchen ein neuartiges Variationsverfahren entwickelt. Wir vergleichen die beiden Methoden bei schwacher und bei starker Kopplung.

3·1. *Schwache Kopplung.* – Aus (19) können wir die ersten Terme einer Entwicklung der exakten Energie des Grundzustandes nach steigenden Potenzen von g ablesen, denn mit dem Kopplungsterm als Störung schreitet die Störungstheorie *systematisch* nach Potenzen von g fort. Das gilt *nicht* für die von Feynman zu diesem Zweck herangezogene Störungstheorie mit

$$\sum (\mathrm{f}, \mathrm{f}') \cdot b_{\mathrm{f}}^{+} b_{\mathrm{f}'}^{+} b_{\mathrm{f}} b_{\mathrm{f}'}$$

als Störung ([9]), die somit auch nicht den « richtigen » g^4-Term liefert ([16]).

Feynmans Variationsverfahren ergibt genau wie auch das Verfahren von Lee und Pines ([14]) für die ersten beiden Glieder:

$$E = -g^2 - \frac{1}{81}\, g^4 = -g^2 - 0.0123 g^4 \,. \tag{20}$$

Unser Ergebnis steht in (16). Allerdings gehört es nicht zur bestmöglichen Funktion F sondern zu einem naheliegenden und einfach durchrechenbaren Ansatz.

3·2. *Starke Kopplung.* – Die Entwicklung des exakten Ausdrucks $E(g)$ nach fallenden Potenzen von g ist von Bogoljubov ([15]) und Tjablikov ([16]) [vgl. auch den Bericht von Yafet ([17])] angegeben worden. Der erste Term stimmt überein mit dem Wert der Energie, der aus Pekars Produktansatz

([13]) R. P. Feynman: *Phys. Rev.*, **97**, 660 (1955).
([14]) T. D. Lee und D. Pines: *Phys. Rev.*, **92**, 883 (1953).
([15]) N. N. Bogoljubov: *Ukrain. Mat. Zhurn.*, **2**, 3 (1950).
([16]) S. W. Tjablikov: *Zhurn. eksper. teor. Fiz.*, **21**, 377 (1951). Deutsche Übersetzung: *Abhandl. Sowj. Phys.*, **4**, 54 (1954).
([17]) Y. Yafet: Techn. Report No. 2. University of Illinois, Urbana (1954).

folgt. Die bisher beste Näherung lautet ([3])

$$E_p = -0.1088 g^4 . \tag{21a}$$

Der zweite Term geht mit g^0 und enthält zwei Anteile

$$-\varkappa^2 \sum \frac{|\varrho_{s0}|^2}{k^2(\varepsilon_s - \varepsilon_0)} - \frac{3}{2} . \tag{21b}$$

Der numerische Wert der ersten Anteils wurde noch nicht ermittelt, wir kommen in **4** darauf zurück und erklären dort auch die Bezeichnungen. Pekars « adiabatische Näherung » ([3]) ergibt in der Ordnung g^4 dasselbe wie der Produktansatz, in der Ordnung g^0 jedoch nur $-3/2$. Sie ist *keine* systematische Entwicklung nach fallenden Potenzen von g. Bemerkenswert ist die einfache Interpretation des 3/2 - Terms: es handelt sich um die Nullpunktsenergie von drei Oszillatorfreiheitsgraden (Energieeinheit $\hbar\omega$), die in die Translationsfreiheitsgrade des Polarons übergegangen sind. — Feynman erhält

$$E = -\frac{1}{3\pi} g^4 - \left(3 \ln 2 + \frac{3}{4}\right) \pm \ldots = -0.106 g^4 - 2.83 \pm \ldots , \tag{22}$$

in der Ordnung g^4 also gerade das Ergebnis des Produktansatzes mit der Näherung (I, 5.13), was offenbar damit zusammenhängt, daß dieses ψ Grundzustand in einem Parabelpotential ist. — Obwohl ein Beweis noch aussteht, darf man wohl annehmen, daß wir mit unserem Ansatz den *exakten* g^4-Term bekommen (21*a*) (*). ϱ und ψ in (7) sind dazu so zu wählen, daß Linearkombinationen (I, 9.3) des besten Produktansatzes entstehen. Aus rechnerischen Gründen haben wir statt dessen in I Linearkombinationen der Näherung (I, 5.13) benutzt. Damit folgt

$$E = -\frac{1}{3\pi} g^4 - \frac{3}{2} - \left(\frac{3}{4}\right)^4 10\pi g^{-4} \pm \ldots . \tag{23}$$

Da wir die Translationsinvarianz berücksichtigt haben, erscheint wie bei Pekar der Summand $-\frac{3}{2}$.

Zusammenfassend stellen wir fest, daß das Feynmansche Verfahren und auch das unsere den gesamten Verlauf der $E(g)$-Kurve qualitativ richtig wiedergeben. In den Grenzfällen schwacher und starker Kopplung stimmen beide etwa gleich gut mit der exakten Kurve überein. Bei mittlerer Kopplung gibt es

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(*) *Anm. bei der Korrektur.* – Der Beweis wurde inzwischen erbracht.

bisher keine Möglichkeit zur Beurteilung des Fehlers, er kann durchaus erheblich größer sein, als Feynman am Schluß seiner Arbeit vermutet. Ein Vorteil des Feynmanschen Verfahrens ist der einfache, eine physikalische Interpretation nahelegende Ansatz. Man könnte etwa versuchen, das fiktive zweite Teilchen mit dem Mittelpunkt des Pekarschen Potentialtopfes zu identifizieren. Unser Verfahren hat dafür einen einfacheren Formalismus, es sind mehr Aussagen möglich, weil wir zugleich eine Näherung für den *Zustandsvektor* des Systems erhalten und weiteren Verbesserungen stehen keine besonderen Schwierigkeiten gegenüber. Es könnte sein, daß Feynmans Ansatz mit einer speziellen Wahl unserer Funktionen F und G äquivalent ist ([8]). Ausgehend von einer Übersetzung von ([13]) in die Sprache der Feynmanschen Operatorenrechnung ([18]) haben wir eine Untersuchung in dieser Richtung begonnen.

4. – Störungstheorie für starke Kopplung.

Wir haben in 3.2 erwähnt, daß die exakte Energie des Grundzustandes bei starker Kopplung nach fallenden Potenzen von g^4 entwickelt werden kann, und daß Pekars Produktansatz auf den ersten Term dieser Reihe führt. Die Differenz gegenüber dem exakten Wert liegt bei *allen* Kopplungsstärken unterhalb einer Schranke und ist schon bei mittlerer Kopplung relativ klein verglichen mit der Energie selbst. Es liegt daher nahe, ein Störungsverfahren zu konstruieren, bei dem Pekars Produktansatz die 0. Näherung darstellt.

Zuvor müssen wir jedoch (I, 5) durch eine von Pekar stammende Überlegung ergänzen. Wir machen einen Produktansatz $\psi\Phi$ und suchen zunächst durch Variation von Φ bei beliebigem, aber festem $\psi(r)$ den besten Gitterzustand Φ_a. Dann bilden wir den Erwartungswert von H mit $\psi\Phi_a$.

$$(\psi\Phi_a, H\psi\Phi_a) = -\varkappa^2 \sum \frac{\varrho^2}{k^2} - (\psi, \Delta\psi)\,,$$

wobei

$$\varrho(k) = \int |\psi(r)|^2 \exp[-i\mathfrak{k}\mathfrak{r}]\,\mathrm{d}\mathfrak{r} \tag{24}$$

ist. Variation von ψ^* mit der Nebenbedingung $(\psi, \psi) = 1$ ergibt für ψ die nichtlineare Gleichung

$$-\Delta\psi + V_0(r)\psi = \varepsilon\psi\,, \tag{25}$$

([18]) R. P. Feynman: *Phys. Rev.*, **84**, 108 (1951).

in der die Abkürzung

$$V_0(r) = -2\varkappa^2 \sum \frac{\varrho(k)}{k^2} \exp[-i\mathfrak{k}\mathfrak{r}] = -2g^2 \int \frac{|\psi(\mathfrak{r}_1)|^2}{|\mathfrak{r}-\mathfrak{r}_1|}\, d\mathfrak{r}_1 \tag{26}$$

benutzt worden ist. Um die physikalische Bedeutung dieses Ausdrucks zu erkennen, bilden wir

$$\Delta V_0(r) = 2\varkappa^2 \sum \varrho \exp[-i\mathfrak{k}\mathfrak{r}] = 8\pi g^2 \psi^* \psi\,. \tag{27}$$

In gewöhnlichen Einheiten lautet diese Gleichung

$$\Delta V_0(r) = 4\pi e^2 \left(\frac{1}{n^2} - \frac{1}{\varepsilon}\right) \psi^* \psi\,.$$

Die Kopplungskonstante haben wir dabei noch durch Materialkonstanten des Kristalls ausgedrückt [2]

$$g^2 = -\frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{\varepsilon}\right)\frac{e^2 u}{\hbar\omega}; \qquad u^2 = \frac{2m\omega}{\hbar}\,. \tag{28}$$

Wir erinnern daran, daß beim Produktansatz (im Widerspruch zur Translationssymmetrie des Problems) ein Punkt des Kristalls ausgezeichnet werden muß. In unsern Formeln ist es der Nullpunkt des Koordinatensystems. Nun lesen wir aus den Gleichungen (25) bis (27) für das durch den Produktansatz angenäherte Polaron die folgende anschauliche Vorstellung ab [3,2]: Die von dem Elektron herrührende Ladungsdichte $e\psi^*\psi$ ist kugelsymmetrisch um den Nullpunkt herum verteilt. Sie polarisiert das Gitter in der Umgebung, ruft also Ionenverschiebungen hervor. Diese erzeugen ein elektrisches Feld, dessen Potential $V_0(r)$ man in der Kontinuumsnäherung leicht berechnen kann (26). Schließlich enthält (25) noch die Forderung, daß das anfangs benutzte ψ zugleich Grundzustand für die Schrödingergleichung mit dem Potential $V_0(r)$ sei. Es handelt sich also um ein self-consistent-field-Problem. Aus rechnerischen Gründen hat Pekar seine Näherungslösung nicht aus der Differentialgleichung bestimmt sondern mit einer direkten Variationsmethode.

Jetzt können wir einen Hamiltonoperator angeben, dessen exakter Grundzustand gleich dem besten Produktzustand ist. Er setzt sich additiv aus dem Elektronenanteil $\mathfrak{p}^2 + V_0(r)$ und dem Gitteranteil $\sum c_\mathfrak{l}^+ c_\mathfrak{l}$ (Vgl. I, 5) zusammen. Wir fügen noch eine Konstante hinzu, damit die Energie E_0 des Grundzustandes von H_0 mit der des besten Produktansatzes

$$E_p = -(\psi, \Delta\psi) - \varkappa^2 \sum \frac{\varrho^2(k)}{k^2}\,, \tag{29}$$

übereinstimmt. Unser ungestörter Hamiltonoperator lautet dann:

$$H_0 = \mathfrak{p}^2 + V_0(r) + \varkappa^2 \sum \frac{\varrho^2(k)}{k^2} + \sum c_{\mathfrak{k}}^+ c_{\mathfrak{k}} \tag{30}$$

Der Störterm ist die Differenz gegenüber dem exakten Hamiltonoperator des Polarons (1), der in unsern Einheiten

$$H = \mathfrak{p}^2 + \sum b_{\mathfrak{k}}^+ b^{\mathfrak{k}} - i\varkappa \sum \frac{1}{k} \left(b_{\mathfrak{k}} \exp\left[i\mathfrak{k}\mathfrak{r}\right] - b_{\mathfrak{k}}^+ \exp\left[-i\mathfrak{k}\mathfrak{r}\right]\right) \tag{31}$$

lautet. Mit $c_{\mathfrak{k}} = b_{\mathfrak{k}} + i\varkappa\varrho/k$ folgt $H - H_0 = H' + H''$

$$\begin{cases} H' = i\varkappa \sum \frac{1}{k} \left\{ c_{\mathfrak{k}} \left(\varrho - \exp\left[i\mathfrak{k}\mathfrak{r}\right]\right) - c_{\mathfrak{k}}^+ \left(\varrho - \exp\left[-i\mathfrak{k}\mathfrak{r}\right]\right)\right\} \\ H'' = -V_0(r) - 2\varkappa^2 \sum \frac{\varrho}{k^2} \exp\left[-i\mathfrak{k}\mathfrak{r}\right] . \end{cases} \tag{32a, b}$$

Da wir vorläufig mit dem besten Produktansatz rechnen, ist wegen (26) $H'' = 0$.

Die Störenergie 1. Näherung verschwindet. Zur Berechnung der 2. Näherung benötigen wir alle Zustände 0. Näherung. Es sind Produkte aus Zuständen des Feldes, die in bekannter Weise mit den Erzeugungsoperatoren $c_{\mathfrak{k}}^+$ aus dem Vakuumzustand Φ_0 ($c_{\mathfrak{k}}\Phi_0 = 0$) aufgebaut werden und Zuständen des Teilchens $\psi_s(\mathfrak{r})$, die als Lösungen des Eigenwertproblems

$$\left[-\Delta + V_0(r) + \varkappa^2 \sum \frac{\varrho^2}{k^2}\right] \psi_s(\mathfrak{r}) = \varepsilon_s \psi_s(\mathfrak{r}) , \tag{33}$$

zu gewinnen sind. Man beachte, daß lediglich die Bestimmung des Grundzustandes, den wir jetzt mit dem Index 0 versehen (ψ_0) ein self-consistent-field-Problem ist. Das so gefundene $V_0(r)$ steht in den Gleichungen mit $s \neq 0$ als *vorgegebenes* Potential und auch ϱ ist stets mit ψ_0 zu bilden. Nichtverschwindende Matrixelemente mit dem Grundzustand gibt es nur, wenn der andere Zustand in seinem Feldanteil genau ein Quant enthält. Es habe den Ausbreitungsvektor $\mathfrak{k}$.

$$\begin{cases} (\mathfrak{k}, s \,|\, H' \,|\, 0) = -\frac{i\varkappa}{k} \int \psi_s^* \psi_0 (\varrho - \exp\left[-i\mathfrak{k}\mathfrak{r}\right]) \, d\mathfrak{r} = \begin{cases} \frac{i\varkappa}{k} \varrho_{s0} & s \neq 0 \\ 0 & s = 0 \end{cases} \\ \varrho_{s0}(\mathfrak{k}) = \int \psi_s^* \psi_0 \exp\left[-i\mathfrak{k}\mathfrak{r}\right] d\mathfrak{r} . \end{cases} \tag{34}$$

Die Energiekorrektur 2. Näherung wird damit

$$E^{(2)} = -\sum_{s\neq 0,\mathfrak{k}} \frac{|(\mathfrak{k}, s\,|\,H'\,|\,0)|^2}{E_{\mathfrak{k},s} - E_0} = -\varkappa^2 \sum \frac{|\varrho_{s0}|^2}{k^2(1+\varepsilon_s-\varepsilon_0)}\,. \tag{35}$$

Um die Abhängigkeit von g besser zu übersehen, führen wir ein [wie YAFET ([17])]:

$$\begin{cases} \mathfrak{r} = \dfrac{\mathfrak{r}'}{g^2}\,, \qquad \mathfrak{k} = g^2\mathfrak{k}', \qquad \mathfrak{p} = g^2\mathfrak{p}', \qquad \varepsilon = g^4\varepsilon'\,, \\ V_0\left(\dfrac{r'}{g^2}\right) = g^4 V_0'(r'), \qquad \psi\left(\dfrac{r'}{g^2}\right) = g^3\psi'(r')\,. \end{cases} \tag{36}$$

Dann kommt g in (33) nicht mehr vor und in (35) bloß noch einmal im Nenner:

$$E^{(2)} = -\frac{1}{2\pi^2}\sum_{s\neq 0} \frac{1}{\varepsilon_s' - \varepsilon_0' + g^{-4}} \int |\varrho_{s0}(\mathfrak{k}')|^2 \frac{\mathrm{d}\mathfrak{k}'}{k'^2}\,. \tag{37}$$

Wir sehen also, daß unsere Störungstheorie bei starker Kopplung eine Korrektur liefert, die von g nicht mehr abhängt, also *sehr klein* gegen den mit g^4 anwachsenden Wert der 0. Näherung wird. Damit ist auf einfache Weise gezeigt, daß Pekars Produktansatz im Grenzfall der starken Kopplung bis auf Terme der Ordnung g^0 auf die exakte *Energie* des Grundzustandes führt. Natürlich folgt daraus nicht unmittelbar etwas über die Güte des Produkt*zustandes*, dieser ist vielmehr schlecht, weil er die Translationsinvarianz des Problems nicht berücksichtigt ([19]). Eine Verbesserung, die diesem Einwand nicht ausgesetzt ist, haben wir in (I, 9.3) angegeben: Geeignete Linearkombinationen von Produktzuständen führen im Grenzfall starker Kopplung zu derselben Energie wie der einzelne Produktzustand und sind außerdem Eigenzustände von $\mathfrak{W}$. Leider kann man nicht so einfach wie hier eine Störungstheorie anschließen.

(37) geht für große g in Bogoljubovs Formel (21*b*) über, nur fehlt der Summand $-\frac{3}{2}$. Das war zu erwarten, wir haben bereits erwähnt, daß dieser mit der Translationsinvarianz zusammenhängt, die in der zu (37) führenden Störungstheorie nicht genügend berücksichtigt ist.

Bisher haben wir vorausgesetzt, daß wir mit dem *besten* Produktansatz

([19]) Falls der Grundzustand nicht entartet ist, muß er Eigenzustand von $\mathfrak{W}$ zum Eigenwert (0, 0, 0) sein. Dann ist er sicher *nicht* lokalisiert. Vgl. H. HAKEN: *Zeits. Naturf.*, **10**a, 253 (1955) sowie I, S. 195. Die Kritik Tjablikovs [([16]), letzte Seite] an FRÖHLICH, PELZER und ZIENAU ([22]) erledigt sich wohl durch die Bemerkung, daß er den Begriff « Autolokalisation » (= self trapping) in anderer Weise benutzt als diese Autoren und auch wir. Tjablikov spricht von Autolokalisation, wenn (25) einen diskreten Eigenwert besitzt. Er behauptet, daß dies bei schwacher Kopplung nicht der Fall sei. Dann müßte die Näherung (40) ungültig werden, denn wenn V_0 weit draußen wie $1/r$ abfällt, gibt es diskrete Eigenwerte.

rechnen. Praktisch gehen wir jedoch von einer Näherung für ψ_0 aus. Wir finden die zugehörigen Näherungen für ϱ, ε_0, $V_0(r)$ aus (24), (29), (33). Das mit diesen Größen nach (30) gebildete H_0 hat als exakten Grundzustand diejenige Näherung zum besten Produktansatz, von der wir ausgegangen sind. H'' verschwindet nicht mehr, es gibt in 2. störungstheoretischer Näherung eine Korrektur zu ε_0, aus der wir entnehmen können, wie weit wir noch von dem zum besten Produktansatz gehörigen Wert entfernt sind.

$$\text{(38)} \qquad \begin{cases} H''_{os} = -\int V_0(r)\psi_0^*\psi_s \,\mathrm{d}\mathfrak{r} - \dfrac{1}{\pi^2}\int \varrho(k)\varrho_{0s}(\mathfrak{k})\,\dfrac{\mathrm{d}\mathfrak{k}}{k^2} \\ \Delta\varepsilon_0 = -\sum\limits_{s\neq 0} \dfrac{|H''_{0s}|^2}{\varepsilon_s - \varepsilon_0}\,. \end{cases}$$

Man könnte z.B. von (I, 5.13) ausgehen.

$$\text{(39)} \qquad \begin{cases} \psi_0'(r') = \left(\dfrac{2}{9\pi^2}\right)^{\frac{3}{4}} \exp\left[-r'^2/9\pi\right], \qquad \varrho'(k') = \exp\left[-9\pi k'^2/8\right], \\ V_0'(r') = \dfrac{4}{81\pi^2}r'^2 - \dfrac{5}{3\pi}, \qquad \varepsilon_0' = -\dfrac{1}{3\pi} = -0.106\,. \end{cases}$$

Die Näherung für $V_0(r)$ ist ziemlich schlecht, denn das Potential muß weit draußen wie $1/r$ abfallen, weil die Ladung lokalisiert ist (27). Pekar erhielt durch Einsetzen von (I, 5.14) in (26) die bisher beste Näherung [3) [$R = 1.3170r'$]

$$\text{(40)} \qquad V_0'(r') = -\frac{2.6340}{R}\Big\{1 - \exp\left[-R\right](1 + 0.7605R + 0.2605R^2 + \\ + 0.05087R^3 + 0{,}005703R^4 + 0.0003024R^5)\Big\}\,.$$

Das Minimum liegt bei -0.631 Energieeinheiten ($g^2\hbar\omega$), der Grundzustand von (25) bei -0.326. Wenn wir die Gitterenergie $(2/\pi)\int \varrho^2\,\mathrm{d}k$ hinzuzählen (33), folgt für den Grundzustand des Polarons -0.1088 (21a). Die Parabel (39) ergibt den beinahe ebenso guten Wert -0.106. Die angeregten Niveaus zeigen jedoch einen großen Unterschied. Bei der Parabel liegen sie äquidistant im Abstand $4/9\pi = 0.141$. Bei dem Potential (40) hingegen liegt der erste angeregte Zustand nur 0.129 Einheiten über dem Grundzustand und die Intervalle werden dann schnell enger. Aus diesem Grunde lohnt sich eine Auswertung von (37) mit (39) nur zum Zwecke des Vergleichs mit dem g^0-Term der Feynmanschen Theorie (22), die ja in enger Beziehung zur Parabelnäherung steht. Die Reihe (37) (ohne g^{-4}) beginnt mit $-3/4 - 9/32$, ..., wurde aber noch nicht aufsummiert (*).

(*) *Anm. bei der Korrektur.* – Herr Müllensiefen hat die Reihe aufsummiert. Das Ergebnis: 3 ln 2 spricht im Hinblick auf (22) für eine anderweitig begründete Vermutung von Allcock, daß in (21b) 3/4 statt 3/2 stehen sollte. Ich danke Herrn Dr. Allcock für eine briefliche Diskussion.

Wir vermuten, daß der genannte Mangel der Näherung (39) auch die Genauigkeit von Feynmans Resultat beeinträchtigt.

5. – Die Masse des Polarons.

Zur Berechnung der Masse gingen Landau und Pekar von einem ruhenden lokalisierten Polaron aus ([3,2]). Sie fanden eine Lösung der klassischen Bewegungsgleichungen der Ionen, die eine gleichförmige Translation dieses Gebildes beschreibt. Die Energie des Systems ist gegenüber dem ruhenden Polaron um einen geschwindigkeitsabhängigen Anteil erhöht, der bei einer Entwicklung nach Potenzen von v mit dem quadratischen Term

$$\frac{M}{2}v^2; \qquad \frac{M}{m}=\frac{2}{3\pi^2}g^8\int\varrho'^2\,\mathrm{d}\mathfrak{k}'=\frac{16\pi}{3}g^8\int|\psi'|^4\,\mathrm{d}\mathfrak{r}', \tag{41}$$

beginnt ([20]). ϱ bzw. ψ' sind aus dem besten Produktansatz zu entnehmen. Wir ergänzen die in (I, 5) angeführten Näherungen durch ([3])

$$\left|\begin{array}{l}\psi(r)\quad (7\pi)^{-\frac{1}{2}}\beta^{-\frac{3}{2}}\left(1-\dfrac{r}{\beta}\right)\exp\left[-r/\beta\right]; \qquad \varrho(k)=\dfrac{1+(\beta k)^2/28}{[1+(\beta k)^2/4]^4}\\[2ex] E_p(g)=-0.1073g^4; \qquad \beta=2g^{-2},\end{array}\right. \tag{42}$$

und stellen in Form einer Tabelle zusammen, welche Werte die einzelnen Näherungen für die Energie des Grundzustandes (29) und für die Masse des Polarons (41) geben.

Tabelle I.

Näherung	$-E\cdot g^{-4}$	$(M/m)\cdot g^{-8}$
(I, 5.14)	0.1088	0.0208
(42)	0.1073	$0.0232 = 437/6\cdot 56^2$
(I, 5.13)	$0.106 = 1/3\pi$	$0.0200 = 16/81\pi^2$
(I, 5.12)	$0.098 = 25/256$	$0.0203 = (1/12)(5/8)^3$

Die Massen sind offenbar viel empfindlicher gegenüber kleinen Abweichungen vom exakten Zustand als die Energien. Für diese ist (I, 5.14) die beste Näherung, da es sich um ein Variationsverfahren handelt.

([20]) Die Formeln (5.41), (5.43), (5.44) des Berichts ([2]) enthalten Druckfehler. Ich bin Herrn Prof. Fröhlich für die Mitteilung der Korrekturen zu Dank verpflichtet.

Formel (41) ist auch das Ergebnis der quantenmechanischen Rechnung Pekars im Rahmen seiner adiabatischen Näherung ([3]). Bogoljubov fand bei der systematischen Entwicklung nach fallenden Potenzen von g ebenfalls (41) als ersten Term ([15-17]). Um für unsern Ansatz die Masse des Polarons zu ermitteln, gehen wir von einer Formel aus, die manchmal als «Theorem von Feynman» ([21]) bezeichnet wird ([22]):

$$\frac{\partial E_j}{\partial \lambda} = \left(\psi_j, \frac{\partial H}{\partial \lambda}\, \psi_j\right), \tag{43}$$

und durch Differentation von $H\psi_j = E_j\psi_j$ zu beweisen ist. Wenn wir für λ unser $\mathfrak{W}'$ einsetzen und für H den Hamiltonoperator (4), so folgt

$$\operatorname{grad}_{\mathfrak{W}'} E(\mathfrak{W}') = 2(\Phi_{\mathfrak{W}'}, [\mathfrak{W}' - \sum \mathfrak{k} b_{\mathfrak{k}}^{+} b_{\mathfrak{k}}]\Phi_{\mathfrak{W}'}) . \tag{44}$$

Nun machen wir die unitäre Transformation (I, 3.2) rückgängig und lesen ab, daß rechts der Erwartungswert des Impulses des Elektrons steht. Der Erwartungswert des Stromes in einem (normierten) Eigenzustand $S\Phi_{\mathfrak{W}'} = \Psi_{\mathfrak{W}'}$ von $\mathfrak{W}$ zum Eigenwert $\mathfrak{W}'$ ist also:

$$(\Psi_{\mathfrak{W}'}, \mathfrak{j}\Psi_{\mathfrak{W}'}) = \frac{e}{m}(\Psi_{\mathfrak{W}'}, \mathfrak{p}\Psi_{\mathfrak{W}'}) = \frac{e}{2m}\operatorname{grad}_{\mathfrak{W}'} E(\mathfrak{W}') . \tag{45}$$

Da E nur von $\mathfrak{W}'^2$ abhängt, beginnt seine Entwicklung

$$E(g, \mathfrak{W}'^2) = E_0(g) + \frac{\mathfrak{W}'^2}{M(g)/m} + \dots . \tag{46}$$

Die Bedeutung des Koeffizienten von $\mathfrak{W}'^2$ wird ersichtlich, wenn wir (46) in (45) einsetzen und zu gewöhnlichen Einheiten übergehen (I, S. 194), $\mathfrak{v} = \mathfrak{p}/m$.

$$M(\Psi_{\mathfrak{W}'}, \mathfrak{v}\Psi_{\mathfrak{W}'}) = \hbar\mathfrak{W}' . \tag{47}$$

Wird nun ein äußeres Feld angelegt, so kann man zeigen, daß $\mathfrak{W}'$ linear mit der Zeit zunimmt, also auch $(\Psi_{\mathfrak{W}'}, \mathfrak{v}\Psi_{\mathfrak{W}'})$, und M/m ist ein Maß dafür, wie stark

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([21]) R. P. Feynman: *Phys. Rev.*, **56**, 340 (1939); S. T. Epstein: *Amer. Journ. Phys.*, **22**, 613 (1954).

([22]) Die folgende Überlegung findet sich bei Bogoljubov ([15]). Auf andere Weise hat H. Haken: *Zeits. Naturf.*, 9a, 228 (1954), die Formel (45) hergeleitet. Für einen speziellen Variationsansatz wurde sie bereits früher von H. Fröhlich, H. Pelzer und S. Zienau bewiesen: *Phil. Mag.*, **41**, 221 (1950); s.a. ([2]).

die Beschleunigung der Ladung durch die Elektron-Gitterwechselwirkung behindert wird.

Die Berechnung der Masse des Polarons ist also gleichbedeutend mit der Bestimmung des Koeffizienten von $\mathfrak{W}'^2$ in einer Entwicklung von $E(\mathfrak{W}'^2)$ nach steigenden Potenzen dieser Größe (46). Wir haben im Anschluß an (I, 12a) diesen Koeffizienten mit (7), (I, 5.13) berechnet. Da die Integrale nicht geschlossen ausgewertet werden konnten, haben wir uns zunächst auf den ersten Term einer Entwicklung nach fallenden Potenzen von g^4 beschränkt:

$$\frac{81\pi^2}{8}\, g^{-8}\mathfrak{W}'^2 \,. \tag{48}$$

Für die Masse des Polarons ergibt sich also

$$\frac{M}{m} = \frac{8}{81\pi^2}\, g^8 + \ldots \,. \tag{49}$$

(48) ist um einen Faktor 2 größer, als man nach den anderen Ergebnissen erwartet. Es ist bemerkenswert, daß der Anteil Gitterenergie + Wechselwirkungsenergie bei uns genau so groß ist, wie bei Pekars halbklassischer Rechnung (41). Nur kommt bei uns schon in der Ordnung g^{-8} ein Anteil von der kinetischen Energie des Elektrons hinzu, der bei Pekar nicht auftritt. Im Hinblick auf diese Schwierigkeit haben wir die numerische Berechnung von $M(g)$ noch nicht ausgeführt, jedoch kann man sehen, daß das Ergebnis für alle Kopplungsstärken qualitativ dem Verlauf der exakten Kurve entspricht.

Feynman hat $M(g)$ bisher nur für die Grenzfälle großer und kleiner g gerechnet. Er fand für schwache Kopplung:

$$\frac{M}{m} = 1 + \frac{g^2}{6} + 0.025 g^4 \,, \tag{50}$$

während für starke Kopplung die Entwicklung wieder mit dem Term beginnt, der zu der Näherung (I, 5.13) des Produktansatzes gehört ([23]) (Tabelle).

Wir erwähnen noch das Ergebnis von LLP

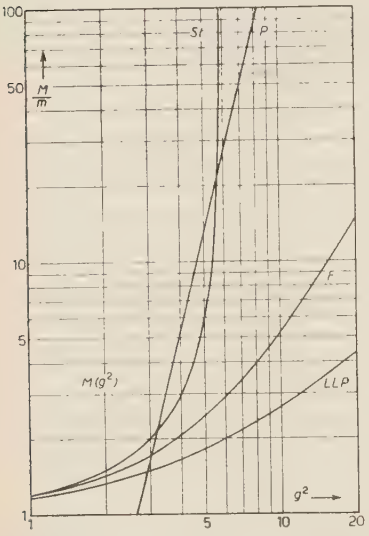
$$\frac{M}{m} = 1 + \frac{g^2}{6} \,. \tag{51}$$

Ihre Störungstheorie gibt eine Korrektur, die bei $g^2 = 5$ etwa mit den nume-

([23]) Wir vermuten, daß Feynmans Formel (47) Druckfehler enthält und richtig heißt: $m = 16\alpha^4/81\pi^2 = 200.2(\alpha/10)^4$. α ist unser g^2.

risch gerechneten Werten von LEE und PINES ([14]) übereinstimmt, dann aber stärker ansteigt. Die Störungstheorie mit dem Kopplungsterm als Störung ergibt in 2. Näherung ([2])

$$\frac{M}{m} = \frac{1}{1 - g^2/6}. \tag{52}$$



Genau dasselbe erhält man auch nach Hagas Methode ([9]) in 2. Näherung ([24]).

Zum Abschluß stellen wir diese Ergebnisse in einem Diagramm zusammen. Der einfachste Verlauf für die exakte Kurve $M(g)$ wäre der folgende: Sie beginnt bei schwacher Kopplung wie (52) geht bei mittlerer Kopplung zwar nicht nach ∞, steigt aber doch außerordentlich steil an (mit g^8 und stärker) und nähert sich dann, wieder etwas flacher werdend, der Landau-Pekarschen Kurve.

Fig. 1. – Masse des Polarons als Funktion der Kopplungsstärke in doppelt-logarithmischer Darstellung. St: Störungstheorie (52), LLP: (51), F: Feynman (50) (Näherung für schwache Kopplung). Lee und Pines ([14]), fanden bei $g^2 = 5{,}2$; 10; 15 die Werte: 2,21; 3,96; 6,35. P: Landau-Pekar (Tabelle, erste Zeile).

* * *

Ich danke Herrn A. MÜLLENSIEFEN für Unterstützung bei den Rechnungen.

([24]) Es ist besser, (4) so aufzuteilen:

$$\mathcal{H}_0 = \mathfrak{W}'^2 + \sum b_{\mathfrak{k}}^+ b_{\mathfrak{k}} (1 \quad 2\mathfrak{k}\mathfrak{W}' + k^2) \quad - i\varkappa \sum (b_{\mathfrak{k}} \quad b_{\mathfrak{k}}^+)/k\,; \quad \mathcal{H}' = \sum (\mathfrak{k}, \mathfrak{k}') b_{\mathfrak{k}}^+ b_{\mathfrak{k}'}^+ b_{\mathfrak{k}} b_{\mathfrak{k}'}\,.$$

Dann folgt (52) schon in 0. Näherung und man kann ohne besondere Schwierigkeiten bis zur 2. Näherung rechnen.

RIASSUNTO (*)

L'impostazione variazionale esaminata in un precedente lavoro presentava bensì nei casi limite di accoppiamento debole e forte il comportamento della soluzione esatta, ma con accoppiamento debole non offriva alcun progresso rispetto all'impostazione di LEE, LOW e PINES. Con l'introduzione di un altro tipo di correlazioni tra i quanti sonori abbiamo ora migliorato la nostra impostazione nel senso che lo stato fondamentale del polarone è descritto circa altrettanto bene come col procedimento di Feynman pubblicato nel frattempo. Si discutono i risultati finora ottenuti per l'energia dello stato fondamentale e la massa del polarone. All'impostazione del prodotto di Pekar si associa un procedimento perturbativo per accoppiamento forte.

(*) *Traduzione a cura della Redazione.*

High-Energy Multiple Photon Production (*).

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(ricevuto il 7 Luglio 1955)

Summary. — Multiple production of photons by fast elementary particles coupled strongly to the electromagnetic field is treated by semiclassical methods. In this approximation, the photons are treated in a precise quantum mechanical fashion, while the motion of the matter field is obtained by classical means but includes radiation reaction effects. As a specific example the magnetic monopole is discussed (and another possible domain of applicability is pointed out). A possible connection with several recent cosmic ray events is investigated. It is shown that conventional electrodynamic models (including antiparticle annihilation) produce too few photons and magnetic monopoles too many, to account for the observed multiplicities.

1. – Introduction.

Considerable speculation was aroused last year by several unusual cosmic ray events reported by groups at Chicago (1) and Torino (2). Although some of these events may possibly be accounted for on the basis of an electromagnetic cascade whose initial photons come from a $\pi^0 \to 2\gamma$ decay, such an explanation would require reasonably large statistical fluctuations. We explore here instead the possibilities of obtaining large photon multiplicities from a single event in the high-energy region both for conventional matter-electromagnetic field couplings and for the more novel effects introduced by considering also

(*) This work was performed under the auspices of the U. S. Atomic Energy Commission while the authors were at the University of California Radiation Laboratory, Berkeley, California, and appeared as UCRL-2747, October 8, 1954.

(1) M. Schein, D. M. Haskin and R. G. Glasser: *Phys. Rev.*, **95**, 855 (1954).

(2) A. Debenedetti, C. M. Garelli, L. Tallone, M. Vigone and G. Wataghin: *Nuovo Cimento* **12**, 954 (1954).

magnetic monopoles. It becomes apparent during the course of these calculations that no such processes can explain the multiplicities encountered in the new events. However, our basic interests lie rather with the investigation on a semi-classical basis of strong-coupling, long-range interactions in the domain of high energies. The discussion is couched in terms of the behavior of monopoles interacting with electrically charged matter which may be of some intrinsic interest. It is possible that at sufficiently high energies the conventional electrodynamics may be usefully viewed in this fashion. As has come to be suspected, the effects of vacuum polarization in shielding the bare electric charge tend to decrease at higher energies as one penetrates through the virtual pair cloud ([3]). Thus in this region, the effective coupling constant for the electromagnetic field may indeed be quite large.

In order to obtain a qualitative idea of the magnitudes to be expected, as well as to exhibit the difficulties in explaining the events, we use numbers of the order given by SCHEIN ([1]). Perhaps the most unique characteristic of the event in comparison with other cosmic-ray phenomena is the occurrence of a very large photon multiplicity, the quanta appearing to emanate from a single near-by point. The extremely narrow angle within which all the photons are found indicates the high energy of the primary involved. Furthermore, despite this, no attendant charged particles were observed, nor were any neutral-particle decays leading back to the original event seen, although a considerable length of emulsion was exposed and scanned by Schein's technique. Thus, any explanation must ensure that the primaries (which interact with the electromagnetic field) be not visible on Schein's plate. The point of origin of the event can be traced back to the vicinity of the aluminium exposure box surrounding the pellicles, indicating the possibility that this material played a role in triggering the event.

It is clear that no calculation based on perturbation theory can be useful in the discussion of such a phenomenon. What is needed is a more rigorous treatment of the coupled-fields problem. Although, of course, such a formalism does not exist, it is possible to treat the boson field rigorously while approximating the matter field by an external current. This would appear to deal with the important aspects of the interactions correctly, as it is the multiplicity of the bosons that is most unusual. Furthermore, although in this approximation the matter field is taken as a prescribed current, radiation reaction effects on it can be included in the calculation of the current by classical means, and indeed are essential at these energies.

([3]) J. SCHWINGER (unpublished); M. GELL-MANN and F. LOW: *Phys. Rev.*, **95**, 1300 (1954).

The particular procedure that we employ (1) yields directly the probability for the production of a given number of photons under the action of any prescribed current. At the same time, (less reliable) information is available as to the energy and angular distributions of the emitted quanta. We may note that this formalism can be used for annihilation as well as for scattering events by a suitable redefinition of the current.

As mentioned previously, it is essential that the orbits of the charged particles involved be calculated in such a way as to include radiation reaction effects. For high-energy phenomena, fortunately, a classical calculation is available, first given by POMERANCHUK (5). In this domain it would seem that quantum effects would be small and such a calculation should be adequate.

2. – Formulation of the Theory.

The theory of multiple photon production (4) has as a consequence that the probability for the emission of n quanta, p_n, by a prescribed current, j_μ, obeys the familiar Poisson distribution,

$$p_n = \frac{W^n}{n!} \exp [-W] , \tag{1}$$

where W is a functional of j_μ given by

$$\begin{cases} W = \frac{1}{2} \int j_\mu(x) D_1(x - x') j_\mu(x') \, \mathrm{d}x \, \mathrm{d}x' , \\ D_1(x) = \frac{1}{(2\pi)^3} \int \exp [ikx] \, \delta(k^2) \, \mathrm{d}^4 k . \end{cases} \tag{2}$$

For sufficiently large W, the most probable number of photons emitted is $n = W$. For small W, the most probable event, of course, is zero photons emitted, higher multiplicities being successively less probable. The dispersion is the $\sqrt{n}$ characteristic of a Poisson distribution.

In the following, it is convenient to represent W as an integral in momentum space. We then have, in general, for the most probable number of emitted

(4) R. GLAUBER: *Phys. Rev.*, **84**, 395 (1951); J. SCHWINGER: *Phys. Rev.*, **91**, 728 (1953); and H. W. LEWIS, J. R. OPPENHEIMER and S. A. WOUTHUYSEN: *Phys. Rev.*, **73** 127 (1948).

(5) I. POMERANCHUK: *Journ. Phys. U.S.S.R.*, **2**, 65 (1940); and L. LANDAU and E. LIFSHITZ: *The Classical Theory of Fields* (1951).

quanta (for large W) an expression of the form

$$n \simeq W = \int n(\theta, k) k^2 \, \mathrm{d}k \, \mathrm{d}\Omega \, . \tag{3}$$

Equation (3) thus furnishes us with a distribution of the quanta in angle and momentum ranges, which may be compared with the observed distribution.

The various production mechanisms may be characterized by the effective current density $j_\mu(x)$ to which they correspond. Since in each case one considers the radiation as being due to the acceleration of (possibly) several charge particles, j_μ has the general form

$$j_{\mu_{\mathrm{tot}}}(x) = \sum_i q_i V_{\mu_i}(t) \delta(r - r_i(t)) \, . \tag{4}$$

The sum extends over the relevant particles; $V_{\mu_i}(t)$ and $r_i(t)$ are the velocity and position of the ith particle, while q_i represents the « charge » on the particle [6].

For the models involving monopoles, the roles of E and H are interchanged. If one considers only the two-field problem (i.e., neglects the coupling to the electron field), the entire formalism outlined above goes through unaltered. Here, however, q_i would represent the monopole's coupling constant.

To determine the orbits for the scattering models to be inserted in Eq. (4), we employ the classical equations of motion for charged particles, including radiation damping:

$$m \frac{\mathrm{d}u_\mu}{\mathrm{d}s} = q F_{\mu\nu} u_\nu + \frac{2}{3} q^2 \left(\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + u_\mu u_\nu \frac{\mathrm{d}^2 u_\nu}{\mathrm{d}s^2} \right) , \tag{5}$$

where $F_{\mu\nu}$ is the external field, $u_\mu = \mathrm{d}x_\mu/\mathrm{d}s$ and s is the proper time. Using a high-energy approximation developed in ref. [5], and assuming rectilinear motion along the x-axis (neglecting deflection for the moment) one obtains [7]

$$\sqrt{1 - V^2(x)} = \sqrt{1 - V_i^2} + \int_{x_0}^{x} \mathrm{d}x \, g(x) \, , \tag{6}$$

where

$$g(x) = \frac{2}{3} m \left(\frac{q^2}{m} \right)^2 \left[(E_y(x) - H_z(x))^2 + (E_z(x) + H_y(x))^2 \right] \tag{7}$$

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[6] Eq. (4) does not include contributions to the current arising from spin moments No detailed consideration of such effects is made in this paper.

[7] $V = \mathrm{d}x/\mathrm{d}t$. The fourth component of the current is of course the charge density Thus, $V_0 = \mathrm{d}t/\mathrm{d}t = 1$.

and V_i is the incident velocity. In our models of the cosmic ray events, the external field is the Coulomb field of an atom. To within desired accuracy, it is there adequate to replace E by a constant of magnitude Ze/r_0^2 over the Fermi-Thomas radius, r_0, and zero outside. Thus

$$g(x) \sim \frac{2}{3} m \left(\frac{q^2}{m}\right)^2 \frac{Z^2 e^2}{r_0^4} . \qquad 0 < x < r_0 \tag{7a}$$

Integrating Eq. (6) gives

$$V(t) = \begin{cases} V_i & t < 0 \\ \cos\,(gt + \cos^{-1} V_i) & 0 < t < t_0 \\ \cos\,(gt_0 + \cos^{-1} V_i) \equiv V_f\,, & t > t_0 \end{cases} \tag{8}$$

where t_0 is the time of traversal ($t_0 \cong r_0/c$) and V_f is the final velocity. Thus $r(t)$ may be obtained by a simple integration of Eq. (8). As we shall see below, W is insensitive to the particular shape of the particle's orbit. The significant information obtained from the above analysis is the final velocity of the particle (i.e., the energy loss) and the amount of deflection it undergoes. In general, at these energies, it will be seen that the following simplified path is adequate to calculate W:

$$V\,(t) = \begin{cases} V_i & t < 0, \\ V_f & t > 0. \end{cases} \tag{9}$$

Turning now to the annihilation models, we note that although the phenomenon of pair annihilation is of quantum origin, it may (for obtaining approximate multiplicities) also be characterized by an effective current. For a fast antiparticle incident upon a stationary particle this current is clearly given by

$$j_\mu = \begin{cases} q[(V_i, 1)\delta(r - V_i t) - (0, 1)\delta(r)] & t < 0, \\ 0 & t > 0, \end{cases} \tag{10}$$

where V_i is the incoming velocity.

Finally, for the annihilation of a fast positronium-like structure, the current takes the form

$$j_\mu = \begin{cases} q[(V_i, 1)\delta(r - V_i t) - (V_i, 1)\delta(r - V_i t - \varrho(t))] & t < 0, \\ 0 & t > 0. \end{cases} \tag{11}$$

Here $\varrho(t)$ is a small distance of the size of the Bohr orbit which goes to zero at $t = 0$; its analytic form may be said to summarize the internal structure of the bound state.

Eqs. (6) and (7) yield the energy loss in a collision. While we shall discuss the specific results for each case later, it is interesting to note the explicit dependence upon the various parameters,

$$\frac{m}{\varepsilon_f} = \frac{m}{\varepsilon_i} + \int_{-\infty}^{\infty} \mathrm{d}x\, g(x) = \frac{m}{\varepsilon_i} + \frac{2}{3}\, m \left(\frac{q^2}{m}\right)^2 \frac{Z^2 e^2}{r_0^3}\,, \tag{12}$$

where ε_i and ε_f are the initial and final energies respectively. For extremely high-energy incident particles the second term on the right-hand side gives a lower limit for the final energy. Because of the strong mass and « charge » dependence appearing in this term, only particles with light mass and (or) large « charge » can radiate appreciably. It is, however, not sufficient for the particle to radiate an amount of energy compatible with Schein's measurements (as, for example, might be achieved by decreasing the impact parameter r_0); the particle must radiate a considerable fraction of its incident energy in order that it be adequately deflected so as not to appear on the plate.

As mentioned above, the details of the path are not relevant in calculating W for collision models. In momentum space, W may be written as

$$W = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2\,|k|} \left| \int \exp\left[-i\,|k|\,(t - n\cdot r(t))\right] V_\mu(t)\, \mathrm{d}t \right|^2, \tag{13}$$

where $n = \boldsymbol{k}/|\boldsymbol{k}|$ and the integration over k_0 has been performed. Integrating once by parts gives

$$W = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2\,|k|^3} \left| \int_0^{t_0} \exp\left[-i\,|k|\,(t - n\cdot r(t))\right] \frac{\mathrm{d}}{\mathrm{d}t} \frac{V_\mu(t)}{1 - n\cdot V(t)}\, \mathrm{d}t \right|^2. \tag{14}$$

In this form, the restriction that radiation will occur only when there is an acceleration is obvious. As is well known, the behavior of the time integration of Eq. (14) is governed by the behavior of the phase. When the latter is very small, the exponential may be placed equal to unity and the integral is seen to depend only on the initial and final velocities. Since, in this case, the current changes rapidly in comparison to the radiated frequency (taking into account the Doppler shift), a discontinuous approximation may be used for the velocity (Eq. (9)). In our case the phase has the order of magnitude

$$kt\left(1 - \frac{n\cdot r(t)}{t}\right) \sim k_{\max} t_0 (1 - V_i)\,, \tag{15}$$

since the radiation is almost entirely in the forward direction. Inserting $k_{max} \sim 10^{12}$ eV (the order of the Schein energies), $t_0 \sim 10^{-19}$ s (the time of transit across an atomic distance) and $1 - V_i \sim 10^{-14}$ (since $E/m \sim 10^7$), the phase is of the order of 10^{-6} radians ([8]).

In the «sudden» approximation, the k integration of. Eq (14) diverges logarithmically at both ends. The low-frequency infinity is the familiar infrared catastrophe that always occurs in this type of problem. As usual, a cutoff is to be inserted corresponding to the lowest observable photon frequency. The ultraviolet divergence is due solely to the use of the sudden approximation. An instantaneous acceleration implies that an infinite energy has been fed into the particle, and is easily remedied by cutting off the integral at the maximum energy available. Had a more realistic path been used, the exponential that we neglected would indeed have furnished such a cutoff.

We conclude this section by noting that in the sudden approximation Eq. (14) becomes ([9])

$$W = \frac{q^2}{4\pi}\frac{1}{\pi}\ln\frac{k_{max}}{k_{min}}\ln\frac{1+V_i}{1-V_i}\frac{1-V_f}{1+V_f}. \tag{16}$$

3. – Conventional Electrodynamic Models.

We now apply the results of the preceding section to various models which remain within the framework of conventional electrodynamics. To begin with, we consider the bremsstrahlung of a fast proton or electron when colliding with an aluminium Coulomb field. In both cases W, the optimal number of photons radiated, is $\lesssim 1$. This may be seen easily be inserting into Eq. (16) the values $q = e$, $k_{max} = 10^{13}$ eV (an extreme upper limit to the energies measured by SCHEIN), $k_{min} = 10^6$ eV (the energy required for a photon to materialize into a pair and hence a lower limit), $1 - V_i \sim 10^{-14}$, and $V_f = 0$ (again as an extreme). Further, for a proton having an impact parameter of the order of a Fermi-Thomas radius (10^{-9} cm), it may easily be seen from Eq. (12) that the energy loss is negligible ($\sim$ keV). It would require an impact parameter $r_0 = 10^{-12}$ cm to obtain energy losses comparable to those observed. Aside from the improbability of such close collisions, the energy loss is so small a

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([8]) Several paths, including the more accurate one in Eq. (8), have actually been integrated approximately; the results in each case corroborate the above argument.

([9]) Numerical factors ~ 1 have been neglected in this formula, which has also been derived by E. CORINALDESI: *Nuovo Cimento*, **12**, 571 (1954). The result for W agrees as well with the quantum mechanical perturbation calculation of S. N. GUPTA: *Phys. Rev.*, **96**, 1453 (1954), who unfortunately omitted a factor of $n!$ in his final results.

fraction of the initial energy (one part in 10^3) that the deflection would be negligible and the particle would certainly have been observed on Schein's plate. Already here and even more so at smaller impact parameters, one would expect some evidence of nuclear interactions (meson production, etc.). For electrons, the energy radiated at the Fermi-Thomas radius is still only $\sim 10^9$ eV. Although it is possible to make the electron radiate $\sim 10^{12}$ eV by reducing the impact parameter (also, thereby, obtaining a larger deflection), the multiplicity $n \lesssim 1$ is so small that this model does not bear serious consideration.

It might be supposed that if the charge on the primary were raised the multiplicity might be adjusted correctly. While this is so for an ion of effective charge 10, this increase is compensated in Eq. (12) by the increase in mass, and the deflection remains much too small.

Finally we consider models based upon a fast antiparticle (positron or antiproton) annihilation. The j_μ for such a process has been given in Eq. (10). Again the current has the same general magnitude as in the scattering models ($q = e$), and a similar calculation for $n \sim W$ confirms the value for multiplicity $\lesssim 1$ ([10]).

Thus, it is clear that in order to obtain both a high multiplicity and large energy loss and deflection it is necessary to postulate a particle with small mass and large effective coupling to the electromagnetic field.

4. – The Magnetic Monopole.

A quantum theory of the magnetic monopole and its interaction with ordinary electrodynamics has been given by DIRAC ([11]). A necessary consequence of the quantization of the electromagnetic field in this theory is the fundamental relation between e and the monopole coupling, g,

$$\frac{eg}{4} = \frac{1}{2}, \qquad \text{or} \qquad g^2/e^2 \sim 5\,000 . \tag{17}$$

In the theory of Dirac, neither of the charged particles is represented by second-quantized fields. Indeed, the difficulty in formulating the general three-field problem lies in the nonexistence of potentials. Of course, either of the two-field interactions can be treated in the usual fashion, the monopole-

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([10]) It should be pointed out that the well-known infrared catastrophe does not remove these difficulties. Any lower cutoff at all consonant with the nature of the experiment yields far too few photons.

([11]) P. A. M. DIRAC: *Phys. Rev.*, **74**, 817 (1948).

electromagnetic system being identical to ordinary electrodynamics with $e \to g$, $F_{\mu\nu} \to F^{+}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F_{\alpha\beta}$. Thus, within this framework, the general boson production formulae hold, j_μ now representing the monopole current. We reserve discussion of the implications of the three-field problem for the next section. Since our proposed model remains within the simpler two-field assumption we proceed with the calculations on the basis of the already developed theory (Section 2).

The energy loss given by Eq. (12) is still valid, as Eq. (7) is invariant under the transformation $F_{\mu\nu} \to F^{+}_{\mu\nu}$. Taking the mass of the pole to be about electronic mass (i.e., $\varepsilon_i \sim 5 \cdot 10^{12}$ eV), we find that almost all the energy has been radiated, i.e., $\varepsilon_f \sim 10^6$ eV (for $r_0 \sim 10^{-9}$ cm). The necessity of this choice of mass becomes clearer upon consideration of the deflection. An adequate idea of its magnitude may be obtained from simple considerations of the momentum acquired in the y direction (p_y) owing to the bending effect of the Coulomb field.

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = q\frac{Ze}{4\pi r_0^2}V_x\,, \qquad p_{y_f} \sim \frac{Zeg}{4\pi r_0}\sqrt{1 - V_f^2}\,. \tag{18}$$

Hence the deflection angle θ is given by

$$\operatorname{tg}\theta = \left(\frac{p_{y_f}}{p_x}\right)_f \sim \frac{Zeg}{4\pi r_0 m} \sim 0.4\,. \tag{19}$$

Examination of the geometry involved in the Schein plates indicates that such a deflection could send the monopole away from the pellicles.

It may be pointed out that all but a small fraction of the energy has been radiated before any appreciable deflection has occurred. (Thus this large angle does not disagree with the observed narrow angle of the shower, and the calculations given below assuming rectilinear motion are adequate). This may be seen qualitatively from the fact that $\operatorname{tg}\theta$ at any point in the collision will have the extra factor of $\sqrt{1 - V^2(t)}$. As soon as this term approaches unity, the energy has been mostly radiated.

We now consider the distribution of emitted quanta in energy and angle $n(\theta, k)$. In calculating W as $\int k^2\,\mathrm{d}k\,\mathrm{d}\Omega\,n(\theta, k)$ we have attempted to roughly take correlations into account, that is to say, the successive emissions are not strictly independent (owing to the requirements of conservation). The derivation of the Poisson distribution neglects this, and we shall to some extent remedy this oversimplification. The effect of the correlations may be divided between the $k^2\mathrm{d}k$ and $\mathrm{d}\Omega$ integrations on the physical grounds that the former should have an upper cutoff (which reflects the fact that no photon will have an excessive energy), while the latter should be restricted to a narrow

forward cone (because of the primary's high forward velocity during emission, as evidenced in the transformation from the c.m. to lab. frame). More explicitly, we considered the available phase space for the n emitted photons ([12]) in the c.m. frame, took the nth root to represent a «mean» photon, and equated the result (while transforming to the lab. frame) to the $\int \mathrm{d}^3k$ of W. The numerical factors appearing in the c.m. phase space (which are due to the energy conservation law) were used to give an energy ($\int k^2\,\mathrm{d}k$) cutoff, while those resulting from the transformation to the lab. system furnished the allowed cone angle. We shall merely quote the result here that $k_{\max} \sim (1/300)\nu$ primary energy, and the cone angle $\theta_m \sim 10^{-3}$ radian. Our answer, then, for the number of quanta emitted below an energy k and within an angle θ is proportional to

$$N(\theta, k) = \int_0^{\theta} \int_{k_{\min}}^{k} \mathrm{d}^3k' n(\theta', k') \sim \ln \frac{k}{k_{\min}} \ln \frac{1 - V_i \cos\theta}{1 - V_i}, \tag{20}$$

where $k_{\min} = 5 \cdot 10^8$ eV (the lowest observed pair energy), $1 - V_i = 10^{-14}$, and $k_{\max} \cong 10^{10}$ eV. [We note that there is a logarithmic dependence in both distributions. The total number ot quanta emitted when correlations are included turns out to be $\sim 10^3$. Thus the monopole is much too strongly coupled to account for the Schein event.

Just as particle-antiparticle bound states exist in ordinary electrodynamics, a structure similar to this may be envisaged in the monopole case. We shall speak of them in analogy to the well-known positronium system. Since the coupling would of course be strong, the similarity between the two structures is to be viewed only in a purely qualitative fashion. One assumes that a fast, stable bound state makes a transition into a state of short lifetime because of interaction with the aluminum's Coulomb field.

Setting (phenomenologically) for the current in Eq. (11) $\varrho(t) = \varrho_0$ for $t < 0$ and zero for $t > 0$, one can find in the usual fashion that the number of photons emitted is ~ 20 for $\varrho_0 \gtrsim 10^{-13}$ cm. Here deflection is no longer a problem.

Unfortunately the significant characteristics of such bound states cannot be calculated in this strong coupling theory. In particular, it is essential to have some idea as to the lifetimes of the states involved, which requires a quantum theoretical investigation. While nothing positive can be stated on this problem, the strong coupling need not imply very short lifetimes. One would expect the decay probability for an annihilation to be proportional to something like $|\psi(0)|^2$. The behavior of wave functions for Coulomb fields with effective coupling constants greater than one have been investigated

([12]) J. V. LEPORE and R. STUART: *Phys. Rev.*, 94, 1724 (1954).

by CASE ([13]). There it was observed that the wave function is highly oscillatory near the origin and hence $|\psi(r)|^2$ may average to a small quantity for small r.

5. – Conclusions.

In this work we have investigated a possible method of dealing with strong-coupling electrodynamic forces. One such example is the theory of the magnetic monopoles. There, the coupling is indeed large and multiple processes are quite favored. It should be remembered, however, that the investigation of monopole phenomena should really be conducted within a three-field framework. The general question concerning the possibility of formulating the full problem along the desired lines is one that cannot be adequately treated because of the lack of a suitable Lagrangian. The two-field approximation employed throughout cannot therefore be validated. It seems likely, however, that if monopoles exist at all, the success of ordinary electrodynamics would weigh in favor of the simple approximation used. A more involved question arises concerning renormalization. If it is assumed that this concept remains valid in the three-field problem, the lack of gauge covariance may imply an absence of Ward's identity. In any event, it remains to be seen how the Dirac condition, $eg/4 = \frac{1}{2}$, is to be interpreted in the light of charge renormalization. Its derivation, of course, is in terms of unrenormalized quantities.

So far as the new cosmic ray events are concerned, it is not surprising that weak-coupling electrodynamics is totally inadequate. On the other hand, the monopole coupling was found to be too strong ([14]). However, as was mentioned earlier, one method of envisaging a considerably stronger coupling than $\alpha = 1/137$ is related to the penetration to the bare charge at high enough energy. On purely heuristic grounds, it is possible to account for the high multiplicity with a particle of charge $q \sim 8e$ and electronic mass. Of course the domain of energy where the effective charge is appreciably increased is not known. Should such energies turn out to be not too excessive, calculations along the lines performed here may prove of some help.

([13]) K. CASE: *Phys. Rev.*, **80**, 797 (1950).

([14]) Throughout this work we have not considered the effect of spin moments on the radiation formulae. These indeed could become appreciable even for convertional electrodynamics at high energies. A detailed knowledge of the moments' form factors would be required to settle this question.

RIASSUNTO (*)

La produzione multipla di fotoni da parte di particelle elementari veloci fortemente accoppiate al campo elettromagnetico si tratta con metodi semiclassici. In questa approssimazione i fotoni sono trattati esattamente dal punto di vista della meccanica quantistica, mentre il moto del campo materiale si ottiene con mezzi classici ma comprende effetti di reazione radiativa. Come esempio specifico si discute il monopolo magnetico (e si indica un altro possibile campo d'applicazione). Si esamina la possibile connessione con alcuni recenti eventi di origine cosmica. Si dimostra che i modelli elettrodinamici convenzionali (compresa l'annichilazione delle antiparticelle) producono un numero insufficiente di fotoni ed eccessivo di monopoli magnetici per render ragione delle molteplicità osservate.

(*) *Traduzione a cura della Redazione.*

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On Conservation Laws in Production and Annihilation of Antinucleons.

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Summary. — The role played by conservation laws and selection rules is analyzed for production and annihilation of antinucleons. The cross sections for these processes are developed in terms of the independent cross sections for the transitions which conserve the total isotopic spin. In addition, the initial and final states are analyzed and selection rules are stated for the different processes.

Recent experimental results that can be interpreted by means of antinucleons [1] and the actual possibility of obtaining protons with energy greater than that necessary for the production of a nucleon-antinucleon pair [2] has aroused interest in antinucleon problems. In this paper we analyze the conservations laws and the selection rules in production and annihilation of antinucleons; a more detailed analysis is carried out at threshold for production, and at low energy of the initial particles for annihilation.

We consider the conservation laws regarding the following operators: J, total angular momentum; U, charge conjugation; P, parity; $W = U \cdot V$, where V is the charge symmetry operator; and T, total isotopic spin.

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[1] M. Schein, D. M. Haskin and R. G. Glasser: *Phys. Rev.*, **96**, 829 (1954); H. S. Bridge, H. Courant, H. DeStaebler and B. Rossi: *Phys. Rev.*, **95**, 1101 (1954); E. Amaldi, C. Castagnoli, G. Cortini, C. Franzinetti and A. Manfredini: *Nuovo Cimento*, **1**, 492 (1955).

[2] Threshold is at 5.6 GeV, kinetic energy of the incident nucleon in the laboratory system. This energy can be at present reached by the Berkeley accelerator.

The selection rules for W and T are weak, being subject to the validity of charge independence.

The eigenvalue of P and T_z assigned to nucleon and antinucleons are the following ([3,4])

	p	$\bar{\text{p}}$	n	$\bar{\text{n}}$
P	+ 1	— 1	+ 1	— 1
T_z	+ 1/2	— 1/2	— 1/2	+ 1/2

In the following we introduce the variable s, that assumes the value 1 for triplet spin states of two nucleons, and the value 0 for singlet states.

1. – Production of Antinucleons.

1·1. *Conservation of isotopic spin.* – In the production of antinucleons by nucleon-nucleon collisions:

$$\mathrm{N} + \mathrm{N} \to \mathrm{N} + \mathrm{N} + \mathrm{N} + \bar{\mathrm{N}} ,$$

the initial states can be developed in eigenstates of T and T_z in the following way:

$$\mathrm{pp} \to \Psi_1^1 ; \qquad \mathrm{pn} \to 1/\sqrt{2}\,\Psi_0^0 + 1/\sqrt{2}\,\Psi_1^0 ; \qquad \mathrm{nn} \to \Psi_1^{-1} .$$

The final states will not only be characterized by the eigenvalues of T, but also by the eigenvalues of the isotopic spin τ of the group of the three nucleons. The possible transitions will therefore be those from the initial wave functions $\Psi_T^{T_z}$ to the final ones ${}_\tau A_T^{T_z}$. The ${}_{\frac{1}{2}}A_1$ and ${}_{\frac{3}{2}}A_1$ are mutually orthogonal, because they belong to different types of symmetry (see M. VERDE ([11])); then no interference term will appear in the total cross-sections: $\sigma_{T,2\tau}$. The cross-sections for the possible physical processes can be described in terms of the three independent σ_{01}, σ_{11} and σ_{13}, with the coefficients given in Table I.

From Table I we can draw some consideration susceptible of eventual experimental check. The contributions of σ_{13} vanish if a deuteron is formed, when possible, among the products of the reaction, it being impossible to have

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([3]) C. N. YANG: *Phys. Rev.*, **77**, 242 (1950); C. N. YANG a J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950) for the eigenvalues of P; the following considerations are also valid for a different choice for the type of the nucleon.

([4]) T. HAMADA and M. SUGAWAKA: *Prog. Theor. Phys.*, **8**, 256 (1952); M. CINI and A. GAMBA: *Nuovo Cimento*, **10**, 1040 (1953) for the eigenvalues of T_z.

TABLE I.

	σ_{01}	σ_{11}	σ_{13}
$p + p \rightleftarrows p + p + p + \bar{p}$	0	0	3/4
$p + p \rightleftarrows p + p + n + \bar{n}$	0	1	1/4
$p + n \rightleftarrows p + n + p + \bar{p}$	1/4	1/4	1/4
$p + n \rightleftarrows p + n + n + \bar{n}$	1/4	1/4	1/4
$n + n \rightleftarrows n + n + p + \bar{p}$	0	1	1/4
$n + n \rightleftarrows n + n + n + \bar{n}$	0	0	3/4

$\tau = \frac{3}{2}$ with a deuteron and a nucleon. In this case all the possible reactions can be expressed be means of only two independent cross-sections (i.e. $\bar{\sigma}_{01}$ and $\bar{\sigma}_{11}$). Further, at threshold, the three nucleons will all tend to be in the s-state with respect to the center of mass system of the final particles. Also, in this case, it will not be possible for a completely antisymmetrized wave function of three nucleons to have $\tau = \frac{3}{2}$, and the contribution of σ_{13} tends to zero when the kinetic energy ω available for the final particles in the C.M. system vanishes.

We have moreover to expect that, at threshold, the production of antineutrons will predominate over that of antiprotons in the p-p collisions, and that the n-p collisions will contribute to the production of antiprotons more than the p-p ones.

1·2. *Conservation of total angular momentum and parity at threshold.* – Let us consider new the collisions p-p and p-n separately, analysing the initial and final states in eigenstates of J and P.

a) $$p + p \rightarrow p + p + p + \bar{p} ;$$

at threshold, one proton must be in a p-state because of the Pauli principle; only the initial states $^1S_0$ and $^1D_2$, with $P = +1$, will contribute to *a*).

b) $$p + p \rightarrow p + p + n + \bar{n} ;$$

all final particles can be in s-state; in this case only the initial states $^3P_0$ and $^3P_1$, with $P = -1$, will contribute to *b*).

c) $$p + n \rightleftarrows \begin{matrix} p + n + p + \bar{p} \\ p + n + n + \bar{n} \end{matrix} ;$$

at threshold, only the initial states $^3P_0$, $^1P_1$ and $^3P_1$, with $P = -1$, will contribute to *c*).

Thus, due to the low energy of the final particles, only a few initial states of the two nucleons can contribute to the production processes.

The antinucleon tends to be emitted in the s-way and then isotropically.

In the processes b) and c) at threshold, the matrix elements for the transitions are independent of the kinetic energy ω at the disposition of the final particles; then the cross-sections will depend on ω only as the statistical factor, i.e. $\sigma \propto \omega^{7/2}$ ([5]). For the reaction a) at threshold, the matrix element will be proportional to $\omega^{1/2}$, so that $\sigma \propto \omega^{9/2}$ ([6]).

2. – Annihilation.

We study the annihilation of a nucleon-antinucleon pair into 2 or 3 pions. In subsection 2·1 we treat the relations, due to total isotopic spin conservation, for the different annihilation processes. In subsection 2·2 we analyze the final states, for each state of the nucleon-antinucleon pair, allowed by conservation of J and P, and by invariance under charge conjugation and symmetry.

2·1. *Conservation of total isotopic spin.* – The initial states can be developed in eigenstates of T and T_z as follows:

$$(1)\qquad \begin{cases} p\bar{p} \to (1/\sqrt{2})\,\Psi_0^0 + (1/\sqrt{2})\,\Psi_1^0\,; & p\bar{n} \to \Psi_1^1\,, \\ n\bar{p} \to \Psi_1^{-1} \qquad\qquad ; & n\bar{n} \to -(1/\sqrt{2})\,\Psi_0^0 + (1/\sqrt{2})\,\Psi_1^0\,. \end{cases}$$

In the two-pion decay:

$$(2)\qquad N + \bar{N} \to \pi + \pi\,,$$

the states of the two mesons that conserve T are Φ_0^0 and $\Phi_1^{T_z}$. The cross sections for the reactions (2) can be obtained as a linear combination of the cross-sections for the reactions $\Psi_i \to \Phi_i$, with the coefficients given in Table II ([7]).

TABLE II.

	σ_0	σ_1
$p + \bar{p} \nearrow \pi^0 + \pi^0$	1/6	0
$p + \bar{p} \searrow \pi^+ + \pi^-$	1/3	1/2
$p + \bar{n} \to \pi^+ + \pi^0$	0	1
$\bar{p} + n \to \pi^- + \pi^0$	0	1
$n + \bar{n} \nearrow \pi^+ + \pi^-$	1/3	1/2
$n + \bar{n} \searrow \pi^0 + \pi^-$	1/6	0

([5]) E. FERMI: *Prog. Theor. Phys.*, **5**, 570 (1950).
([6]) D. FOX: *Phys. Rev.*, **94**, 499 (1950); R. N. THORN: *Phys. Rev.*, **94**, 501 (1954).
([7]) Matrix elements will obviously be independent of T_z.

From Table II we can infer that the process $p + \bar{p} \to \pi^+ + \pi^-$ will predominate over $p + \bar{p} \to \pi^0 + \pi^0$. If it is possible to measure the cross-sections for these two processes, it will be possible to derive σ_0 and σ_1 and then to deduce cross-sections for the other processes.

For the decay into three pions:

$$N + \bar{N} \to \pi + \pi + \pi \tag{3}$$

the states of the three mesons are not only characterized by the total isotopic spin T, but they can be formed in several ways. These ways are characterized by the symmetry properties, of the final eigenfunctions (see Appendix) with the following schemes:

the first of them corresponding to $T = 0$ and the others to $T = 1$.

Let σ be the cross sections for the transitions corresponding to the different final symmetry. The cross-sections for all the reactions (3) can be expressed as combination of $\sigma_{\square\square\square}$, $\sigma_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ and $\sigma_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$, with the coefficients given in Table III [8].

As we do not know the relations between $\sigma_{\square\square\square}$, $\sigma_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ and $\sigma_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ we can not infer anything about the relative yields of charged and neutral pion annihilations of $p\bar{p}$ and $n\bar{n}$. For the cases of $n\bar{p}$ and $p\bar{n}$ the decay into states with greatest number of charged mesons is preferred.

TABLE III.

	$\sigma_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$	$\sigma_{\square\square\square}$	$\sigma_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$
$p + \bar{p} \to \pi^0 + \pi^0 + \pi^0$	0	3/10	0
$p + \bar{p} \to \pi^+ + \pi^- + \pi^0$	1/2	1/5	1/2
$p + \bar{n} \to \pi^+ + \pi^- + \pi^+$	0	4/5	1/2
$p + \bar{n} \to \pi^+ + \pi^0 + \pi^0$	0	1/5	1/2
$p + \bar{n} \to \pi^- + \pi^+ + \pi^-$	0	4/5	1/2
$p + \bar{n} \to \pi^- + \pi^0 + \pi^0$	0	1/5	1/2
$n + \bar{n} \to \pi^0 + \pi^0 + \pi^0$	0	3/10	0
$n + \bar{n} \to \pi^+ + \pi^- + \pi^0$	1/2	1/5	1/2

[8] As shown in Appendix, interference terms are not present.

2·2. *Invariance under charge conjugation and charge symmetry.* – Let $a_{p,n}$, $b_{p,n}$ and $c_{+,0,-}$ be the destruction operators of nucleons, antinucleons and pions, respectively. Their complex conjugates are the creation operators.

Let U be the unitary charge conjugation operator defined by [9]:

$$U\Psi U^{-1}=C\overline{\Psi}\ ; \qquad U\overline{\Psi}U^{-1}=C^{-1}\Psi\ ;$$

$$U\Phi U^{-1}=\Phi^*\ ; \qquad U\Phi^*U^{-1}=\Phi \quad \text{and} \quad U\Phi_3U^{-1}=\Phi_3\ ,$$

where Ψ and Φ are the nucleon and meson wave functions, respectively, and $\overline{\Psi}=\Psi^*\gamma_4$.

The destruction operators satisfy the relations:

$$Ua_{p,n}U^{-1}=b_{p,n}; \qquad Ub_{p,n}U^{-1}=a_{p,n}; \qquad Uc_{+,-}U^{-1}=c_{-,+} \qquad \text{and} \qquad Uc_0U^{-1}=c_0\ ,$$

and similar ones are satisfied by creation operators.

For pseudoscalar mesons, the interaction hamiltonian is invariant under charge conjugation both for *ps* and *pr* coupling. The states $p\overline{p}$ and $n\overline{n}$ are eigenstates of U with eigenvalues $(-1)^{l+s}$, where l is the angular momentum of the system.

Any state formed by an arbitrary number of pions and with total charge zero is also an eigenstate of U; in particular, $\pi^+ + \pi^- - n\pi^0$ is an eigenstate of U with eigenvalue $(-1)^L$, where L is the relative angular momentum of the $\pi^+\pi^-$ system.

Let V be the unitary charge symmetry operator, defined by:

$$V\Psi V^{-1}=\tau_1\Psi\ , \qquad V\overline{\Psi}V^{-1}=\overline{\Psi}\tau_1\ ,$$

$$V\Phi V^{-1}=\Phi^*\ , \qquad V\Phi^*V^{-1}=\Phi \quad \text{and} \quad V\Phi_3V^{-1}=-\Phi_3\ .$$

The destruction operators satisfy the relations:

$$Va_{p,n}V^{-1}=a_{n,p}; \qquad Vb_{p,n}V^{-1}=b_{n,p}\ ; \qquad Vc_{+,-}V^{-1}=c_{-,+} \qquad \text{and} \qquad Vc_0V^{-1}=-c_0\ ;$$

the creation operators satisfy similar relations.

The Kemmer symmetric interaction is invariant under charge symmetry.

Let us define $W=U\cdot V$. The states $p\overline{n}$ and $n\overline{p}$ are eigenstates of W, with eigenvalues $(-1)^{l+s}$; besides, any state formed by an arbitrary number of mesons is an eigenstates of W, with eigenvalues $(-1)^{n_{\pi^0}}$, where n_{π^0} is the number of the natural pions that it contains.

[9] L. WOLFENSTEIN and D. G. RAVENHALL: *Phys. Rev.*, **88**, 279 (1952); A. PAIS and R. JOST: *Phys. Rev.*, **87**, 871 (1952); L. MICHEL: *Nuovo Cimento*, **10**, 319 (1953).

In Table IV are listed the eigenvalues of U, W and P of the physical states we are interested in. In Table IV, V and VI we shall indicate as L, for the states of three mesons, the relative angular momentum of the first two written, and with l' the angular momentum of the third with respect to the C.M. of the first two ([10]).

TABLE IV.

	$p\bar{p}$; $\bar{n}n$	$n\bar{p}$; $p\bar{n}$	$2\pi^0$	$\pi^+\pi^-$	$\pi^0\pi^+$; $\pi^0\pi^-$
U	$(-1)^{l+s}$	—	$+1$	$(-1)^l$	—
W	—	$(-1)^{l+s}$	$+1$	$+1$	-1
P	$(-1)^{l+1}$	$(-1)^{l+1}$	$(-1)^l$	$(-1)^l$	$(-1)^l$
		$\pi^0\pi^0\pi^+$		$\pi^+\pi^+\pi^-$	
	$3\pi^0$	$\pi^0\pi^0\pi^-$	$\pi^+\pi^-\pi^0$	$\pi^-\pi^-\pi^+$	
U	$+1$	—	$(-1)^L$	—	
W	-1	$+1$	-1	$+1$	
P	$(-1)^{L+l'+1}$	$(-1)^{L+l'+1}$	$(-1)^{L+l'+1}$	$(-1)^{L+l'+1}$	

The conservation of the eigenvalues listed in Table IV and the symmetrization of the eigenfunction corresponding to mesons of equal charge imposes the following relations among the angular momenta of the particles in the different processes:

$$
(4)\qquad
\begin{cases}
p\bar{p};\ n\bar{n} \to \pi^0\pi^0 : & s=1;\quad l_i \text{ odd},\ l_f \text{ even}\\
\to \pi^+\pi^- : & s=1;\quad l_i + l_f \text{ odd}\\
\to \pi^0\pi^0\pi^0 : & l_i + s \text{ even};\quad L \text{ even};\quad l_i + l' \text{ even}\\
\to \pi^+\pi^-\pi^0 : & l' + s \text{ even};\quad l_i + L + l' \text{ even}\\
p\bar{n};\ n\bar{p} \to \pi^\pm\pi^0 : & l_i + s \text{ odd};\quad l_i + l_f \text{ odd}\\
\to \left.\begin{matrix}\pi^0\pi^0\pi^\pm\\ \pi^+\pi^+\pi^-\\ \pi^-\pi^-\pi^+\end{matrix}\right\} : & l_i + s \text{ even};\quad L \text{ even};\quad l_i + l' \text{ even}.
\end{cases}
$$

where l_i is the initial orbital momentum and l_f, for the decay into two mesons, the final one.

([10]) For the states of three neutral pions this representation does not take into account the complete symmetrization of the eigenfunctions. To do this a metod similar to that used by G. RACAH: *Phys. Rev.*, **63**, 367 (1943), should be used. Then some of the states of three π^0 allowed in Table V will not be physically possible.

TABLE V. – $\bar{p}p$ *and* $\bar{n}n$.

	U	P	$\pi^0\pi^0$	$\pi^+\pi^-$	$\pi^0\pi^0\pi^0$	$\pi^+\pi^-\pi^0$
$^1S_0$	+	−	−	−	$(Ss)_0$, $(Dd)_0$, ...	$(Ss)_0$, $(Dd)_0$, ...
$^3S_1$	−	−	−	P_1	−	$(Pp)_1$, $(Ff)_1$, ...
$^1P_1$	−	+	−	−	−	$(Ps)_1$, $(Pd)_1$, $(Fd)_1$, ...
$^3P_0$	+	+	S_0	S_0	−	−
$^3P_1$	+	+	−	−	$(Sp)_1$, $(Dp)_1$, $(Df)_1$, ...	$(Sp)_1$, $(Dp)_1$, $(Df)_1$, ...
$^3P_2$	+	+	D_2	D_2	$(Pp)_2$, $(Df)_2$, $(Ef)_2$, ...	$(Dp)_2$, $(Df)_2$, $(Ef)_2$, ...
$^1D_2$	+	−	−	−	$(Ds)_2$, $(Dd)_2$, ...	$(Ds)_2$, $(Dd)_2$, ...
$^3D_1$	−	−	−	P_1	−	$(Pp)_1$, $(Ff)_1$, ...
$^3D_2$	−	−	−	−	−	$(Pp)_2$, $(Pf)_2$, ...

In Table V and VI are listed the decays allowed for the different states of the nucleon-antinucleon pair, as follows from (4) and conservation of J.

TABLE VI. – $p\bar{n}$ *and* $n\bar{p}$.

	W	P	$\pi^\pm\pi^0$	$\pi^+\pi^+\pi^-$: $\pi^-\pi^-\pi^+$: $\pi^0\pi^0\pi^\pm$
$^1S_0$	+	−	——	$(Ss)_0$, $(Dd)_0$, ...
$^3S_1$	−	−	P_1	− − −
$^1P_1$	−	+	——	− − −
$^3P_0$	+	+	− − −	——
$^3P_1$	+	+	——	$(Sp)_1$, $(Dp)_1$, $(Df)_1$, ...
$^3P_2$	+	+	− − −	$(Dp)_2$, $(Df)_2$, ...
$^1D_2$	+	−	——	$(Sd)_2$, $(Ds)_2$, $(Dd)_2$, ...
$^3D_1$	−	−	P_1	− − −
$^3D_2$	−	−	——	− − −

The states of three mesons are characterized by the values of $(Ll')_J$. Dotted lines indicate weakly forbidden decays, which means that they involve conservation of the eigenvalues of W.

The selection rules listed in Tables V and VI rule out many of the possible final states. But while they are really important in the two meson decay, the high energy available in the three meson decay, makes it possible to have high angular momenta and thus a great number of states.

Any singlet state of the $N - \bar{N}$ pair cannot decay into pions; experimental data on the annihilation at rest would eventually provide some conclusions on the kind of interaction between nucleon and antinucleon.

As can be seen from Table VI the $\bar{p}n$ and $\bar{n}p$ systems, at rest, can decay into two or three mesons, according as they are in triplet or singlet state, respectively. That would suggest some analogy between the $^3S_1$ and $^1S_0$ states of $N - \bar{N}$, and the heavy unstable mesons χ and τ; the angular momentum

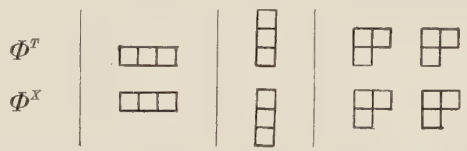
and the parity of the state $^1S_0$ ($J=0$; $P=-1$) are the same as the spin and the parity at present attributed to the τ-mesons.

We are pleased to thank Proff. E. AMALDI and L. A. RADICATI for useful discussions and criticism.

APPENDIX.

The final states will be of the form: $\Phi = \Phi^T \cdot \Phi^X$, where Φ^T is the isotopic spin wave function and x indicates every other variable.

We need transitions to an orthogonal set of Φ^X, in order that the integrations performed to calculate total cross-sections eliminate interference terms. Φ must be totally symmetric, i.e. of the type $\square\square\square$. Φ^T and Φ^X can have symmetries characterized by the Young symbols: $\square\square$, $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ and $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ [11]. Therefore, to obtain a totally symmetric Φ we must combine:



The different representations for the Φ^X form an orthogonal set; we can define then transitions to the following representations for the Φ^T: $\square\square\square$, $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ and $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, doing no distinction between the two configurations of the type $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ or any linear combination of them, because they always give the same contribution.

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[11] H. WEYL: *The Theory of Groups and Quantum Mechanics*; A. GAMBA: *Rev. Scient.*, **90**, 11 (1952); M. VERDE: *Helv. Phys. Acta*, **22**, 339 (1949).

RIASSUNTO

Si studia il ruolo delle leggi di conservazione e delle relative regole di selezione nei processi di produzione e di annichilamento degli antinucleoni. Le sezioni d'urto per questi processi sono sviluppate come combinazione lineare delle sezioni d'urto indipendenti per le transizioni che conservano lo spin isotopico totale. Inoltre si analizzano gli stati iniziali e finali e si ottengono le regole di selezione per i differenti processi.

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Over-all Space-time Description and Third Quantization.

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Summary. — In order to reformulate quantum field theory from the over-all space-time point of view, we take up the Feynman amplitudes constructed by the second quantized operators. In treating these Feynman amplitudes, the third quantized operators which create or annihilate the second quantized operators are introduced. These third quantized operators are embedded in a new Hilbert space and operate upon the state vectors in this space. The connection between this formulation and the usual second quantized theory is investigated. It is shown that our formalism is equivalent to Coester's theory. Finally, employing the third quantization formalism, we discuss a possibility of further extension of the over-all space-time description.

1. – Introduction.

The field theory, founded on the quantum mechanics and the relativity theory, has been formulated in an elegant way by Tomonaga, Schwinger, Feynman and Dyson. However, their theory, called by the name of the covariant field theory, retains several problematic points. Some of them are related to the canonical formalism and the others to the renormalization theory.

It seems natural to construct the covariant theory on the canonical formalism, as it is connected with our usual intuition in which all phenomena develop with time. But there seems for us to exist the following questionable points. In this formalism the existence of the total energy-momentum four vector of a system as well as that of its minimum value are usually required. Although the concept of the energy-momentum four vector does not seem contradictory to the relativistic covariance at first sight, this is not always evident. For example, if one takes the vacuum state as the minimum state of a system.

the momentum and the energy do not form a four vector in this state. Actually, one has to do with the minimum value of energy or the Hamiltonian in a coordinate system of zero momentum whereas the covariant formalism demands the minimum value of the energy-momentum four vector. Rigorously speaking the conventional Hamiltonian theory is not covariant, unless the minimum energy happens to be zero.

Moreover, it is doubtful whether we need to assume the existence of the Hamiltonian or not. In fact, the Hamiltonian can be defined only in limited cases, where one can never escape from the divergence difficulties which are inherent in the conventional field theory, as clarified by KÄLLÉN [1] and LEHMANN [2]. It might be possible to avoid their criticism only in a very fortunate case, though we think it too optimistic. At any rate, if one wants consciously to overcome the divergence difficulties, it would be necessary to consider the theory of non-Hamiltonian systems. One example of such a theory is non-local theory and the other is non-linear theory.

In non-local theory, several authors have once believed in the existence of the Hamiltonian [3], but its existence is not generally proved [4]. Moreover, it is not yet clear if we can overcome the divergence in this theory. Indeed, KÄLLÉN and MØLLER [5] have expressed a negative point of view. Since their assumptions are too severe, however, we may still retain non-local theory as one of the possibilities.

The situation is more complicated in non-linear theory. Although we try to obtain a consistent theory, perhaps it would be difficult to construct the Hamiltonian in an easy-going way. Probably, in order to realize Heisenberg's idea [6], it may happen that we are obliged to deny the possibility of the existence of the Hamiltonian.

Then we are led to examine the reformulation of the field theory from a wider point of view including a non-Hamiltonian system. For this purpose, it will be convenient to take up the over-all space-time point of view which was already considered by many authors [7].

This stand-point is also connected with the second problem of the covariant field theory. This is related to such a picture of the elementary particles that is based on the renormalization theory, which is successful to separate the

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[1] G. KÄLLÉN: *Helv. Phys. Acta*, **25**, 417 (1952); *Dan. Mat. Fys. Medd.*, **27**, N. 12 (1953).

[2] H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

[3] W. PAULI: *Nuovo Cimento*, **10**, 648 (1953); Y. KATAYAMA: *Prog. Theor. Phys.*, **10**, 31 (1953).

[4] C. HAYASHI: *Prog. Theor. Phys.*, **10**, 533 (1953).

[5] C. MØLLER: *Proc. Intern. Conf., Kyoto, Japan* (1953).

[6] W. HEISENBERG: *Nachr. Akad. Wiss. Göttingen*, n. 8 (1953).

[7] For instance, K. SYMANZIK: *Zeits. f. Naturf.*, **10**a, 809 (1954).

divergence, tentative as it is. In this procedure the divergences are subtracted step by step using the perturbation method. The elementary particle is, in its nature, considered as being surrounded by the cloud of other particles, which causes the divergence; however, even if one could define an elementary particle surrounded by some cloud before an interaction, it would not be guaranteed that one could find exactly the same cloud after the interaction. Namely, the cloud of an elementary particle could change from time to time. Consequently, the identification of an elementary particle depends upon the time of observation. The clouded elementary particle can be defined, only if its over-all motion is pursued from the very beginning to the end. That is, we can accept the definition of the elementary particle only from the over-all space-time stand-point. When one wants to describe the time development of a system in the conventional theory, there are two alternative ways. Either one defines an elementary particle at infinite past, or one defines it at infinite future. As we do not like to accept the former way we should reformulate the theory by taking up the latter one more positively. Therefore, if we could construct the pre-renormalized theory, by which we mean that the renormalization effects are taken into account automatically from the beginning, there being no need of further performing the renormalization procedure, it should be founded on the theory developed on the over-all space-time point of view.

In order to formulate the theory from the over-all space-time view, we have to keep in mind the following points. Firstly, the most appropriate way for us is to pick up from the conventional covariant field theory the quantities which are suitable to this stand-point and must bear the essential features of that theory as the starting point of the new theory. We share with other authors the belief [8] that such quantities are Feynman amplitudes. The reason why we pick up them is motivated by the fact that, starting from the over-all space-time point of view, Feynman succeeded to derive the same conclusions as those of the covariant field theory. Feynman amplitudes do not depend on the specific time and are connected with other amplitudes, according to the properties of the second quantized field operators contained. In order that such properties are also retained in the new formalism, it is necessary to have some method to describe the transition between the amplitudes. Such a kinematical description must imply the conventional quantization procedure, but need not contain all its details; that is, to describe such a transition we rest upon only spins and statistics, without using the knowledge of the masses and the commutation relations of respective field quantities. Such a method

[8] P. T. MATTHEWS and A. SALAM: *Proc. Roy. Soc.*, A **221**, 128 (1953); K. NISHIJIMA: *Prog. Theor. Phys.*, **10**, 549 (1953).

was already developed by NAMBU [9] and was named by him « the third quantization method », because it contains the operators which create or annihilate the second quantized field operators.

The third quantized operators satisfy the four-dimenionally infinitesimal commutation relations depending only on their spins and statistics. Such commutation relations have been used by many authors [10] in the form of the functional derivatives.

In order to complete this formalism, it is convenient to introduce state vectors in a new Hilbert space, as Nambu has tried. By use of these vectors, we can rewrite in accordance with our formalism the Feynman amplitudes given by conventional field operators.

Next, to obtain the dynamical description, it is necessary to know the detailed properties of the Feynman amplitudes, and, for such purpose, we have to use the conventional quantization method. But this may be admitted if the obtained results match with the over-all space-time point of view. Thus we arrive at the dynamical description using the correspondence between the state vectors in the second quantized Hilbert space and those in the third quantized Hilbert space. This method results in the same conclusions as Coester's [11] theory. Some obscure points with respect to the properties of the state vectors in Coester's theory, because he founded his theory on the Matthews-Salam equation, are now clarified in our theory.

The third quantization formalism formulated in this paper is not satisfactory, because there is no clear way for defining the eigenstate. Though we are trying to overcome this defect, we can not yet succeed to find a satisfactory answer.

At any rate it may be inferred that, if one stands on the over-all space-time view, one has to take into account many kinds of elementary particles as well as those having essentially different properties (for example, elementary particles with negative-energy) besides ordinary ones. At the same time, it may happen to be necessary to include the elementary particles with several different masses.

2. – State Vectors in Quantum Field Theory.

If we take the system of a fermion interacting with a neutral boson, the state vector in the Heisenberg representation can be expanded as

$$(1) \qquad \Psi_s = \sum_{n,m,k} (n!m!k!)^{-1} \int \mathrm{d}^3x_n \dots \mathrm{d}^3x_1 \int \mathrm{d}^3y_1 \dots \mathrm{d}^3y_m \int \mathrm{d}^3z_1 \dots \mathrm{d}^3z_k \cdot$$

$$\cdot f_s(x_n \dots x_1; y_1 \dots y_m; z_1 \dots z_k) N_{\mathrm{H}}(z_k \dots z_1; y_m \dots y_1; x_1 \dots x_n) \Psi_0 ,$$

(9) Y. NAMBU: *Prog. Theor. Phys.*, **4**, 331, 399 (1949).
(10) For instance, K. YAMAZAKI: *Prog. Theor. Phys.*, **7**, 449 (1952).
(11) F. COESTER: *Phys. Rev.*, **95**, 1318 (1954).

where Ψ_0 represents the true vacuum state. $N_{\mathrm{H}}(z_k \ldots; y_m \ldots; x_1 \ldots)$ denotes the normal product of the Heisenberg operators of fermions, $\psi_{\mathrm{H}}(x_i)$, $\bar{\psi}_{\mathrm{H}}(y_j)$ and of bosons, $\varphi_{\mathrm{H}}(z_r)$, such as $N_{\mathrm{H}}(\varphi_{\mathrm{H}}(z_k) \ldots; \bar{\psi}_{\mathrm{H}}(y_m) \ldots; \psi_{\mathrm{H}}(x_1) \ldots)$. However, the normal product of the Heisenberg operators is not well-defined usually. Then we tentatively follow the definition of MATTHEWS and SALAM ([8]), namely

$$(2)\qquad N_{\mathrm{H}}(z_k \ldots z_1; y_m \ldots y_1; x_1 \ldots x_n) = T_{\mathrm{H}}(z_k \ldots z_1; y_m \ldots y_1; x_1 \ldots x_n) - \\ - \sum_{j,i} S_{\mathrm{F}}(y_j - x_i) T_{\mathrm{H}}(z_k \ldots z_1; y_j^{-1}, y_m \ldots y_1; x_i^{-1} x_1 \ldots x_n) - \\ - \sum_{s<t} \Delta_{\mathrm{F}}(z_s - z_t) T_{\mathrm{H}}(z_s^{-1}, z_t^{-1}, z_k \ldots z_1; y_m \ldots y_1; x_1 \ldots x_n) + \ldots ,$$

where « T » means the Wick's T-product.

In eq. (1) we take the times of the operators to be the same. The reason is based on the fact that the above bases of the expansion of our state vectors always form a complete set at one time and they successively change into other complete sets at subsequent moments.

Now, the system of these bases is not an orthogonal one, though it is complete. $f_s(x_n \ldots x_1; y_1 \ldots y_m; z_1 \ldots z_k)$'s were called contravariant components by NISHIJIMA ([8]) by the analogy of the oblique coordinate system in geometry, he further introduced the covariant components:

$$(3)\qquad g_s(x_n \ldots x_1; y_1 \ldots y_m; z_1 \ldots z_k) = \langle \Psi_0, N_{\mathrm{H}}(x_n \ldots x_1; y_1 \ldots y_m; z_1 \ldots z_k) \Psi_s \rangle ,$$

which are the so-called Feynman amplitudes. The ortho-normalization of state vectors is expressed by

$$(4)\qquad \langle \Psi_r, \Psi_s \rangle = \sum_{\mathcal{N}} \int f_r^*(\mathcal{N}) g_s(\mathcal{N}) = \delta(r, s) , \qquad \mathcal{N} \equiv (x_n \ldots; y_1 \ldots; z_1 \ldots)$$

and the matrix element of any operator Q is given by

$$(5)\qquad \langle \Psi_r, Q \Psi_s \rangle = \sum_{\mathcal{N}, \mathcal{N}'} \int g_r^*(\mathcal{N}') Q(\mathcal{N}', \mathcal{N}) f_s(\mathcal{N}) .$$

In these formulae $\int$ denotes the spatial integration.

The connection between the contravariant and the covariant components are given by Feynman kernels, though in general it is rather complicated. One can however be informed of the state vectors by the analysis only of covariant components.

In obtaining the covariant component (i.e. Feynman amplitude) from eq. (2), we need to know

$$(6)\qquad \langle \Psi_0, T_{\mathrm{H}}(x_n \ldots x_1; y_1 \ldots y_m; z_1 \ldots z_k) \Psi_s \rangle ,$$

or

$$(7)\quad \begin{cases} \langle \Psi_0,\, T_{\mathrm{H}}(\psi_{\mathrm{H}}(x_n)\dots\psi_{\mathrm{H}}(x_1)\overline{\psi}_{\mathrm{H}}(y_1)\dots\overline{\psi}_{\mathrm{H}}(y_m)\varphi_{\mathrm{H}}(z_1)\dots\varphi_{\mathrm{H}}(z_k))\Psi_s\rangle = \\ \quad = \langle \Phi_0,\, T(U\psi(x_n)\dots\psi(x_1)\overline{\psi}(y_1)\dots\overline{\psi}(y_m)\varphi(z_1)\dots\varphi(z_k))\Phi_s\rangle \cdot \\ \quad \cdot \langle \Phi_0,\, T(U)\Phi_0\rangle^{-1}, \\ U = \exp\left[i\int \mathrm{d}^4x : L_1(\psi(x),\, \overline{\psi}(x),\, \varphi(x)) :\right]. \end{cases}$$

where Φ_s and $\psi(x_i)$, $\overline{\psi}(y_j)$, $\varphi(z_r)$ are the state vector and the operators in the interaction representation, and : : means the normal product. (6) is called the Salpeter-Bethe amplitude.

Originally, both the Feynman amplitudes and Salpeter-Bethe amplitudes have the over-all space-time characters and are independent of special reference systems. Therefore, it seems unfavorable to obtain the contravariant components at a special reference system using such complicated procedures, in order to see the behaviour of the system.

It is desirable to take another way, in which the procedures become simpler and are independent of any special reference system. Indeed, the aim of this paper is to show one of these possibilities.

3. – The Third Quantization Operators.

The first step in the above stated program is to find an appropriate method of treating the transition between Feynman amplitudes. The transition between Feynman amplitudes is caused by the creation or annihilation of Heisenberg operators in the normal product. This is analogous to the creation or annihilation of the number of quanta in the second-quantization formalism.

Following this analogy we can construct the new formalism by treating the creation or annihilation of the number of operators. This is the so-called « third quantization » method developed by NAMBU.

For simplicity, let us consider the Feynman amplitudes which contain only neutral boson operators

$$(8)\qquad g_n(z_1\dots z_n) = \langle \Psi_0,\, N_{\mathrm{H}}(\varphi_{\mathrm{H}}(z_1)\dots\varphi_{\mathrm{H}}(z_n))\Psi\rangle\,,$$

and construct a column matrix

$$(9)\qquad \begin{Vmatrix} g_0 \\ g_1(z_1) \\ g_2(z_1, z_2) \\ \vdots \\ g_n(z_1\dots z_n) \\ \vdots \end{Vmatrix} = \{g_n(z_1\dots z_n)\}\,,$$

which is the representation of a state vector.

Now we introduce the operators $C(z)$ and $C^{+}(z)$, which have the following properties ([12])

$$C(z)\{g_n(z_1 \dots z_n)\} = \{(n+1)^{\frac{1}{2}} g_{n+1}(z_1 \dots z_n, z)\}, \tag{10}$$

$$C^{+}(z)\{g_n(z_1 \dots z_n)\} = \{n^{\frac{1}{2}} \mathfrak{S}_n \delta(z - z_n) g_{n-1}(z_1 \dots z_{n-1})\} . \tag{11}$$

$\mathfrak{S}_n F(z_1 \dots z_n)$ denotes the symmetric part of the function $F(z_1 \dots z_n)$ obtained by dividing by $n!$ the sum of the functions which results by all permutations of $z_1, \dots, z_n$ from $F(z_1 \dots z_n)$. It reflects the symmetrical property of the Feynman amplitudes consisting of boson operators. The matrix representations of these relations are

$$\begin{pmatrix} 0 & \int dz_1' \delta(z - z_1') & 0 & \cdot \\ 0 & 0 & 2^{\frac{1}{2}} \int dz_1' dz_2' \delta(z_1 - z_1') \delta(z - z_2') & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} g_0 \\ g_1(z_1') \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} g_1(z) \\ 2^{\frac{1}{2}} g(z_1, z) \\ \cdot \\ \cdot \end{pmatrix} \tag{10'}$$

and

$$\begin{pmatrix} 0 & 0 & \cdot & \cdot \\ \mathfrak{S}_1 \delta(z_1 - z) & 0 & \cdot & \cdot \\ 0 & 2^{\frac{1}{2}} \mathfrak{S}_2 \int dz_1' \delta(z_1 - z_1') \delta(z_2 - z) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} g_0 \\ g_1(z_1') \\ g_2(z_1', z_2') \\ \cdot \end{pmatrix} = \tag{11'}$$

$$= \begin{pmatrix} 0 \\ \mathfrak{S}_1 \delta(z - z_1) g_0 \\ 2^{\frac{1}{2}} \mathfrak{S}_2 \delta(z - z_2) g_1(z_1) \\ \cdot \end{pmatrix} .$$

From (10) and (11) we derive the relation

$$C^{+}(z') C(z'') \{g_n(z_1 \dots z_n)\} = \{n \mathfrak{S}_n \delta(z' - z_n) g_n(z_1 \dots z_{n-1}, z'')\}. \tag{12}$$

Here also $\mathfrak{S}_n$ indicates that the function is to be symmetrized with respect to $z_1, \dots, z_n$, but not to z' and z''. On the other hand one deduces the relation.

([12]) K. O. FRIEDRICHS: *Mathematical Aspects of the Quantum Theory of Fields* (New York, 1953).

immediately from (10) and (11),

$$(13)\quad C(z'')C^{+}(z')\{g_n(z_1 \dots z_n)\} = \{\delta(z'-z'')g_n(z_1 \dots z_n) + n\mathfrak{S}_n\delta(z'-z_n)g_n(z_1 \dots z_{n-1}, z'')\}.$$

Subtracting (12) from (13) we find

$$(14a)\qquad [C(z'')C^{+}(z') - C^{+}(z')C(z'')]\{g_n(z_1 \dots z_n)\} = \delta(z'-z'')\{g_n(z_1 \dots z_n)\}\,.$$

Similarly we have the obvious relations

$$(14b)\qquad [C(z'')C(z') - C(z')C(z'')]\{g_n(z_1 \dots z_n)\} = 0\,,$$

$$(14c)\qquad [C^{+}(z'')C^{+}(z') - C^{+}(z')C^{+}(z'')]\{g_n(z_1 \dots z_n)\} = 0\,.$$

For the Feynman amplitude of fermion operators ψ_{H}'s

$$(15)\qquad g_n(x_1 \dots x_n) = \langle \Psi_0, N_{\mathrm{H}}(\psi_{\mathrm{H}}(x_1) \dots \psi_{\mathrm{H}}(x_n))\Psi\rangle\,,$$

we introduce the following operators

$$(16)\qquad A(x)\{g_n(x_1 \dots x_n)\} = \{(n+1)^{\frac{1}{2}}g_{n+1}(x_1 \dots x_n, x)\}\,,$$

$$(17)\qquad A^{+}(x)\{g_n(x_1 \dots x_n)\} = \{n^{\frac{1}{2}}\mathfrak{A}_n\delta(x-x_n)g_{n-1}(x_1 \dots x_{n-1})\}\,.$$

$\mathfrak{A}_n$ indicates the anti-symmetrical permutation of $x_1, \dots, x_n$, divided by $n!$. In complete analogy to the above derivation of (14), we have

$$(18a)\qquad [A(x'')A^{+}(x') + A^{+}(x')A(x'')]\{g_n(x_1 \dots x_n)\} = \delta(x'-x'')\{g_n(x_1 \dots x_n)\}\,,$$

$$(18b)\qquad [A(x'')A(x') + A(x')A(x'')]\{g_n(x_1 \dots x_n)\} = 0\,,$$

$$(18c)\qquad [A^{+}(x'')A^{+}(x') + A^{+}(x')A^{+}(x'')]\{g_n(x_1 \dots x_n)\} = 0\,.$$

Also for the Feynman amplitudes of operators $\bar{\psi}_{\mathrm{H}}$'s

$$(19)\qquad g_n(y_1 \dots y_n) = \langle \Psi_0, N_{\mathrm{H}}(\bar{\psi}_{\mathrm{H}}(y_1) \dots \bar{\psi}_{\mathrm{H}}(y_n))\Psi\rangle\,,$$

$B(y)$ and $B^{+}(y)$ are introduced as

$$(20)\qquad B(y)\{g_n(y_1 \dots y_n)\} = \{(n+1)^{\frac{1}{2}}g_{n+1}(y_1 \dots y_n, y)\},$$

$$(21)\qquad B^{+}(y)\{g_n(y_1 \dots y_n)\} = \{n^{\frac{1}{2}}\mathfrak{A}_n\delta(y-y_n)g_{n-1}(y_1 \dots y_{n-1})\}\,,$$

and we have

$$(22a)\quad [B(y'')\ B^{+}(y') + B^{+}(y')B(y'')]\ \{g_n(y_1 \dots y_n)\} = \delta(y'-y'')\{g_n(y_1 \dots y_n)\}\ ,$$

$$(22b)\quad [B(y'')B(y') + B(y')\ B(y'')]\ \{g_n(y_1 \dots y_n)\} = 0\ ,$$

$$(22c)\quad [B^{+}(y'')B^{+}(y') + B^{+}(y')B^{+}(y'')]\{g_n(y_1 \dots y_n)\} = 0\ .$$

A's and B's are anticommutable with each other. The matrix representations are given by the similar forms to (10′) and (11′).

4. – Hilbert Space in the Third Quantization Formalism.

To construct the third quantization formalism, we introduce the new Hilbert space — we call it the Ω-space — which is the analogy with the Hilbert space in the second quantization formalism. Then we assume that the state vector in this Hilbert space can be expanded by the base vectors with the Feynman amplitudes as the coefficients of that expansion, e.g.:

$$(23)\quad \boldsymbol{\Omega}_{\mathrm{H}} = \sum_{n,m,k} (n\,!m\,!k\,!)^{-\frac{1}{2}} \int \mathrm{d}^4x_n \dots \mathrm{d}^4x_1 \int \mathrm{d}^4y_1 \dots \mathrm{d}^4y_m \int \mathrm{d}^4z_1 \dots \mathrm{d}^4z_k \cdot$$
$$\cdot g_{nmk}(x_n \dots x_1;\ y_1 \dots y_m;\ z_1 \dots z_k)\boldsymbol{\Omega}_{nmk}\ ,$$

$$(24)\quad \boldsymbol{\Omega}_{nmk} \equiv \boldsymbol{\Omega}_{nmk}(z_k \dots z_1;\ y_m \dots y_1;\ x_1 \dots x_n)\ .$$

$\boldsymbol{\Omega}_{nmk}$ constitute an ortho-normal complete set in the Ω-space as follows,

$$(25)\quad \langle \boldsymbol{\Omega}_{nmk}(z'_k \dots z'_1;\ y'_m \dots y'_1;\ x'_1 \dots x'_n),\ \boldsymbol{\Omega}_{nmk}(z_k \dots z_1;\ y_m \dots y_1;\ x_1 \dots x_n)\rangle =$$
$$= (n\,!\ m\,!\ k\,!)^{-1} \sum_{s\neq t}^{k} \sum_{i\neq j}^{m} \sum_{u\neq v}^{n} \dots \varepsilon^{uv\dots}_{ij\dots}\ \delta(z_k - z'_s) \dots \delta(z_1 - z'_t)\cdot$$
$$\cdot\ \delta(y_m - y'_j) \dots \delta(y_1 - y'_i)\delta(x_1 - x'_u) \dots \delta(x_n - x'_v)\ ,$$

$$\langle \boldsymbol{\Omega}_{nmk},\ \boldsymbol{\Omega}_{n'm'k'}\rangle = 0, \qquad \text{for } n \neq n',\ \text{or } m \neq m',\ \text{or } k \neq k',$$

where $\varepsilon^{uv\dots}_{ij\dots}$ is the sign function determined by the totally anti-symmetrical properties of $\boldsymbol{\Omega}_{nmk}$ with respect to y's and x's. Then the norm of the state vector $\boldsymbol{\Omega}_{\mathrm{H}}$ is given by

$$(26)\quad \langle \boldsymbol{\Omega}_{\mathrm{H}},\ \boldsymbol{\Omega}_{\mathrm{H}}\rangle = \sum_{n,m,k} \int \mathrm{d}^4x_n \dots \mathrm{d}^4z_k\, |g_{nmk}(x_n \dots z_k)|^2\ ,$$

and obviously we have

$$(27)\qquad \mathbf{\Omega}_{nmk}(z_k \dots z_1; y_m \dots y_1; x_1 \dots x_n), \mathbf{\Omega}_{\mathrm{H}}) = \\ = (n!\, m!\, k!)^{-\frac{1}{2}} g_{nmk}(x_n \dots x_1; y_1 \dots y_m; z_1 \dots z_k).$$

In order to fulfill the ortho-normalization condition (25), we introduce the creation and annihilation operators, which satisfy the commutation relations

$$(28)\qquad [\mathfrak{c}(z''), \mathfrak{c}^+(z')] = \delta(z'-z''), \qquad [\mathfrak{c}(z''), \mathfrak{c}(z')] = [\mathfrak{c}^+(z''), \mathfrak{c}^+(z')] = 0\,,$$

$$(29)\qquad [\mathfrak{a}(x''), \mathfrak{a}^+(x')] = \delta(x'-x''), \qquad [\mathfrak{a}(x''), \mathfrak{a}(x')] = [\mathfrak{a}^+(x''), \mathfrak{a}^+(x')] = 0\,,$$

$$(30)\qquad [\mathfrak{b}(y''), \mathfrak{b}^+(y')] = \delta(y'-y''), \qquad [\mathfrak{b}(y''), \mathfrak{b}(y')] = [\mathfrak{b}^+(y''), \mathfrak{b}^+(y')] = 0\,,$$

and we construct $\mathbf{\Omega}_{nmk}$ by

$$(31)\qquad \mathbf{\Omega}_{nmk} = (n!\, m!\, k!)^{-\frac{1}{2}} \mathfrak{c}^+(z_k) \dots \mathfrak{c}^+(z_1)\mathfrak{b}^+(y_m) \dots \mathfrak{b}^+(y_1)\mathfrak{a}^+(x_1) \dots \mathfrak{a}^+(x_n)\mathbf{\Omega}_0\,,$$

where $\mathbf{\Omega}_0$ is the state vector subject to:

$$(32)\qquad \mathfrak{a}(x)\mathbf{\Omega}_0 = 0\,, \qquad \mathfrak{b}(y)\mathbf{\Omega}_0 = 0\,, \qquad \mathfrak{c}(z)\mathbf{\Omega}_0 = 0\,.$$

The correspondence between the above operators and the operators introduced in the foregoing chapter can be established precisely as [13]

[13] The meaning of this correspondence is as follows: The matrix has the properties represented in (10′) and (11′), or more compactly written as

$$\langle (k+1)^{\frac{1}{2}} \int \dots \int \mathrm{d}z'_1 \dots \mathrm{d}z'_{k+1} \delta(z_1 - z'_1) \dots \delta(z_k - z'_k)\delta(z - z'_{k+1}) \rangle_{k,k+1} \langle g_{k+1}(z'_1 \dots z'_{k+1}) \rangle_{k+1} = \\ = \langle (k+1)^{\frac{1}{2}} g_{k+1}(z_1 \dots z_k, z) \rangle_k\,,$$

$$\langle k^{\frac{1}{2}} \mathfrak{S}_k \int \dots \int \mathrm{d}z'_1 \dots \mathrm{d}z'_{k-1} \delta(z_1 - z'_1) \dots \delta(z_{k-1} - z'_{k-1})\delta(z_k - z) \rangle_{k,k-1} \langle g_{k-1}(z'_1 \dots z'_{k-1}) \rangle_{k-1} = \\ = \langle k^{\frac{1}{2}} \mathfrak{S}_k \delta(z - z_k) g_{k-1}(z_1 \dots z_{k-1}) \rangle_k\,,$$

while the matrices $\langle \mathbf{\Omega}_k, \mathfrak{c}\ \mathbf{\Omega}_{k+1} \rangle$ and $\langle \mathbf{\Omega}_k, \mathfrak{c}^+ \mathbf{\Omega}_{k-1} \rangle$ operate on $g_k(z_1 \dots z_k)$ such as:

$$\left(\int \mathrm{d}z'_1 \dots \mathrm{d}z'_{k+1} \langle \mathbf{\Omega}_k(z_1 \dots z_k), \mathfrak{c}(z) \mathbf{\Omega}_{k+1}(z'_1 \dots z'_{k+1}) \rangle g_{k+1}(z'_1 \dots z'_{k+1}) \right)_k = ((k+1)^{\frac{1}{2}} g_{k+1}(z_1 \dots z_k, z))_k\,,$$

$$\left(\int \mathrm{d}z'_1 \dots \mathrm{d}z'_{k-1} \langle \mathbf{\Omega}_k(z_1 \dots z_k), \mathfrak{c}^+(z) \mathbf{\Omega}_{k-1}(z_1 \dots z'_{k-1}) \rangle g_{k-1}(z'_1 \dots z'_{k-1}) \right)_k = \\ = (k^{\frac{1}{2}} \mathfrak{S}_k \delta(z - z_k) g_{k-1}(z_1 \dots z_{k-1}))_k\,.$$

Therefore the correspondence is evident.

$$
(33)\qquad \begin{cases} (C(z))_{k,k+1} \to \langle \mathbf{\Omega}_{nmk},\ \mathfrak{c}(z)\mathbf{\Omega}_{nmk+1}\rangle\ , \\ (C^{+}(z))_{k+1,k} \to \langle \mathbf{\Omega}_{nmk+1},\ \mathfrak{c}^{+}(z)\mathbf{\Omega}_{nmk}\rangle\ , \\ (A(x))_{n,n+1} \to \langle \mathbf{\Omega}_{nmk},\ \mathfrak{a}(x)\mathbf{\Omega}_{n+1mk}\rangle\ , \\ (A^{+}(x))_{n+1,n} \to \langle \mathbf{\Omega}_{n+1mk},\ \mathfrak{a}^{+}(x)\mathbf{\Omega}_{nmk}\rangle\ , \\ (B(y))_{m,m+1} \to \langle \mathbf{\Omega}_{nmk},\ \mathfrak{b}(y)\mathbf{\Omega}_{nm+1k}\rangle\ , \\ (B^{+}(y))_{m+1,m} \to \langle \mathbf{\Omega}_{nm+1k},\ \mathfrak{b}^{+}(y)\mathbf{\Omega}_{nmk}\rangle\ , \end{cases}
$$

with the same algebras (28)-(30) and (14), (18) and (22).

If we utilize the relation (31), we have

$$
(34)\qquad \begin{cases} \int \mathrm{d}z\ \mathfrak{c}^{+}(z)\ \mathfrak{c}(z)\ \mathbf{\Omega}_{nmk} = k\mathbf{\Omega}_{nmk}\ , \\ \int \mathrm{d}x \mathfrak{a}^{+}(x)\mathfrak{a}(x)\mathbf{\Omega}_{nmk} = n\mathbf{\Omega}_{nmk}\ , \\ \int \mathrm{d}y \mathfrak{b}^{+}(y)\mathfrak{b}(y)\mathbf{\Omega}_{nmk} = m\mathbf{\Omega}_{nmk}\ , \end{cases}
$$

and find that $\mathbf{\Omega}_{nmk}$ are the simultaneous eigenvectors of the operators:

$$
(35)\qquad \begin{cases} \mathfrak{N}_c = \int \mathrm{d}z\ \mathfrak{c}^{+}(z)\ \mathfrak{c}(z)\ , \\ \mathfrak{N}_a = \int \mathrm{d}x \mathfrak{a}^{+}(x)\mathfrak{a}(x)\ , \\ \mathfrak{N}_b = \int \mathrm{d}y \mathfrak{b}^{+}(y)\mathfrak{b}(y)\ . \end{cases}
$$

and have their eigenvalues k, n and m respectively. Similarly $\mathfrak{c}\mathbf{\Omega}_{nmk}$ and $\mathfrak{c}^{+}\mathbf{\Omega}_{nmk}$ correspond to eigenvalues $k-1$ and $k+1$ respectively. Therefore we can see that $\mathfrak{c}$'s are the annihilation operators and $\mathfrak{c}^{+}$s are the creation operators in this Hilbert space, while for the Feynman amplitude C's create the numbers of operators in the normal product and C^{+}'s annihilate the numbers of operators.

5. – Connection between two Hilbert Spaces and Coester's Theory.

In order to establish the correspondence between the state vectors in the usual Hilbert space of the second quantization and those in our Hilbert space of the third quantization, it is most convenient to take up the Feynman amplitudes.

Using (27), (31) and the definition (3) of the Feynman amplitudes we can easily derive the relation:

$$\langle \Psi_0, N_{\mathrm{H}}(x_n \dots x_1; y_1 \dots y_m; z_1 \dots z_k)\Psi\rangle = \\ = \langle \mathbf{\Omega}_0, \mathfrak{a}(x_n) \dots \mathfrak{a}(x_1)\mathfrak{b}(y_1) \dots \mathfrak{b}(y_m)\mathfrak{c}(z_1) \dots \mathfrak{c}(z_k)\mathbf{\Omega}\rangle \,. \tag{36}$$

To satisfy (36) there must be some connection between Ψ and $\mathbf{\Omega}$, and indeed we can see that the equality

$$\mathbf{\Omega}_{\mathrm{H}} = \langle \Psi_0, N_{\mathrm{H}}\left(\exp\left[\int \mathrm{d}x(\mathfrak{a}^+\psi_{\mathrm{H}} - \bar{\psi}_{\mathrm{H}}\mathfrak{b}^+ + \varphi_{\mathrm{H}}\mathfrak{c}^+)\right]\right)\Psi\rangle\mathbf{\Omega}_0 \,, \tag{37}$$

holds. Thus there is a definite $\mathbf{\Omega}_{\mathrm{H}}$ corresponding to the given Ψ. To make this fact more clear, let us consider the similar state vector $\mathbf{\Omega}$ corresponding to the state vector Φ in the interaction representation:

$$\mathbf{\Omega} = \langle \Phi_0, N\left(\exp\left[\int \mathrm{d}x(\mathfrak{a}^+\psi - \bar{\psi}\mathfrak{b}^+ + \varphi\mathfrak{c}^+)\right]\right)\Phi\rangle\mathbf{\Omega}_0 \,. \tag{38}$$

ψ, $\bar{\psi}$ and φ are the operators in the interaction representation, and so they satisfy

$$\left(\gamma\frac{\partial}{\partial x} + \varkappa\right)\psi(x) = 0 \,,$$

$$\left(\gamma^T\frac{\partial}{\partial x} - \varkappa\right)\bar{\psi}(x) = 0 \,,$$

$$(\Box_x - \mu^2)\varphi(x) = 0 \,.$$

Owing to these properties, in the exponent of (38) there remain solely the operators $\mathfrak{a}^+$, $\mathfrak{b}^+$ and $\mathfrak{c}^+$ which satisfy the energy momentum relations of free particles. This fact is responsible for the fact that $\mathfrak{a}^+$, $\mathfrak{b}^+$ and $\mathfrak{c}^+$ are essentially equivalent to the interaction representation operators, in spite of their having many degrees of freedom.

This can also be seen from the fact that, if we define $\mathbf{\Omega}$ by (38), we naturally obtain the relation:

$$\begin{cases} \left(\gamma\dfrac{\partial}{\partial x} - \varkappa\right)\mathfrak{a}(x)\mathbf{\Omega} = 0 \,, \\ \left(\gamma^T\dfrac{\partial}{\partial x} - \varkappa\right)\mathfrak{b}(x)\mathbf{\Omega} = 0 \,, \\ (\Box_x - \mu^2)\mathfrak{c}(x)\mathbf{\Omega} = 0 \,. \end{cases} \tag{39}$$

In Coester's theory the equations in (39) are the conditions imposed on $\mathbf{\Omega}_{\mathrm{H}}$, yet in our treatment we can show that these are deduced by using the correspondence relation (38).

From Matthews and Salam's definition (2) of the « N » product (37) and (38) are rewritten as:

$$(40)\quad \mathbf{\Omega}_{\mathrm{H}} = \exp\left[\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{+}(x)s_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y) - \frac{1}{2}\iint \mathrm{d}z\,\mathrm{d}z'\,\mathfrak{c}^{+}(z)\,\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{+}(z')\right]\cdot$$
$$\cdot\langle \Psi_0,\, T_{\mathrm{H}}\left(\exp\left[\int \mathrm{d}x(\mathfrak{a}^{+}\psi_{\mathrm{H}} - \bar{\psi}_{\mathrm{H}}\mathfrak{b}^{+} + \varphi_{\mathrm{H}}\mathfrak{c}^{+})\right]\right)\Psi\rangle\mathbf{\Omega}_0\,,$$

$$(41)\quad \mathbf{\Omega} = \exp\left[\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{-}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y) - \frac{1}{2}\iint \mathrm{d}z\,\mathrm{d}z'\,\mathfrak{c}^{+}(z)\,\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{+}(z')\right]\cdot$$
$$\cdot\langle \Phi_0,\, T\left(\exp\left[\int \mathrm{d}x(\mathfrak{a}^{+}\psi - \bar{\psi}\mathfrak{b}^{+} + \varphi\,\mathfrak{c}^{+})\right]\right)\Phi\rangle\mathbf{\Omega}_0\,.$$

From (40) we derive

$$\langle \Psi_0,\, T_{\mathrm{H}}(x_n \dots x_1;\, y_1 \dots y_m;\, z_1 \dots z_k)\Psi\rangle = \langle \mathbf{\Omega}_0,\, \mathfrak{a}(x_n)\dots\mathfrak{a}(x_1)\mathfrak{b}(y_1)\dots\mathfrak{b}(y_m)\mathfrak{c}(z_1)\dots\mathfrak{c}(z_k)\cdot$$
$$\cdot\exp\left[-\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{+}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y) + \frac{1}{2}\iint \mathrm{d}z\,\mathrm{d}z'\,\mathfrak{c}^{+}(z)\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{+}(z')\right]\mathbf{\Omega}\quad,$$

and using (32) and the relations,

$$(42a)\quad \exp\left[+\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{-}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{-}(y)\right]\mathfrak{a}(x)\cdot$$
$$\cdot\exp\left[-\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{+}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y)\right] = \mathfrak{a}(x) - \int \mathrm{d}y\,S_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y) \qquad \boldsymbol{\psi}(x)\,,$$

$$(42b)\quad \exp\left[+\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{+}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{+}(y)\right]\mathfrak{b}(y)\cdot$$
$$\cdot\exp\left[-\iint \mathrm{d}x\,\mathrm{d}y\,\mathfrak{a}^{+}(x)S_{\mathrm{F}}(x-y)\mathfrak{b}^{-}(y)\right] = \mathfrak{b}(y) - \int \mathrm{d}x\,\mathfrak{a}^{+}(x)S_{\mathrm{F}}(x-y) \cdot \bar{\boldsymbol{\psi}}(y)\,.$$

$$(42c)\quad \exp\left[-\frac{1}{2}\iint \mathrm{d}z\,\mathrm{d}z'\,\mathfrak{c}^{+}(z)\,\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{+}(z')\right]\mathfrak{c}(z)\cdot$$
$$\cdot\exp\left[-\frac{1}{2}\iint \mathrm{d}z\,\mathrm{d}z'\,\mathfrak{c}^{+}(z)\,\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{+}(z')\right] = \mathfrak{c}(z) - \int \mathrm{d}z'\,\Delta_{\mathrm{F}}(z-z')\,\mathfrak{c}^{-}(z') \cdot \boldsymbol{\varphi}(z)\,,$$

we finally get

$$(43)\quad \langle \Psi_0,\, T_{\mathrm{H}}(x_n \dots x_1;\, y_1 \dots y_m;\, z_1 \dots z_k)\Psi\rangle =$$
$$= \langle \mathbf{\Omega}_0,\, \boldsymbol{\psi}(x_n)\dots\boldsymbol{\psi}(x_1)\bar{\boldsymbol{\psi}}(y_1)\dots\bar{\boldsymbol{\psi}}(y_m)\boldsymbol{\varphi}(z_1)\dots\boldsymbol{\varphi}(z_k)\mathbf{\Omega}_{\mathrm{H}}\rangle\,.$$

In an entirely analogous way we find:

$$(44) \qquad \langle \Phi_0, T(x_n \dots x_1; y_1 \dots y_m; z_1 \dots z_k)\Phi \rangle = \\ = \langle \Omega_0, \psi(x_n) \dots \psi(x_1)\bar{\psi}(y_1) \dots \bar{\psi}(y_m)\varphi(z_1) \dots \varphi(z_k)\Omega \quad .$$

Now, by making use of (44) the right hand side of (6) can be written as

$$\langle \Omega_0, \psi(x_n) \dots \bar{\psi}(y_1) \dots \varphi(z_1) \dots \exp\left[i\int \mathrm{d}x\, [: L_I(\psi, \bar{\psi}, \varphi): - L_0]\right] \Omega \quad .$$

On the other hand, noticing that the left hand side of (6) is given by (43) we obtain

$$\Omega_{\mathrm{H}} = \exp\left[i\int \mathrm{d}x [: L_I(\psi, \bar{\psi}, \varphi): - L_0]\right] \Omega .$$

This relation can also be obtained directly from (40) and (41) by making use of (6) and (7), which are the equality connecting interaction representation and Heisenberg representation.

The fact that (45) satisfies the relations:

$$(45) \qquad \begin{cases} \left[\left(\gamma^\mu \dfrac{\partial}{\partial x} + \varkappa\right) \mathfrak{a}(x) - \dfrac{\delta \bar{L}_I(\psi, \bar{\psi}, \varphi)}{\delta \psi(x)}\right] \Omega_{\mathrm{H}} = 0 , \\ \left[\left(\gamma^\mu \dfrac{\partial}{\partial x} - \varkappa\right) \mathfrak{b}(x) - \dfrac{\delta \bar{L}_I(\psi, \bar{\psi}, \varphi)}{\delta \bar{\psi}(x)}\right] \Omega_{\mathrm{H}} = 0 , \\ \left[(\Box_x - \mu^2)\, \mathfrak{c}(x) + \dfrac{\delta \bar{L}_I(\psi, \bar{\psi}, \varphi)}{\delta \varphi(x)}\right] \Omega_{\mathrm{H}} = 0 , \end{cases}$$

can be deduced from (39) and (45) or also from (37).

6. – Further Extension of the Formalism.

In the previous sections we have discussed the connection between the second and the third quantization formalisms. It consists in the correspondence (37) between the state vectors in both formalisms, and if we accept this correspondence, then we can automatically derive every character of the third quantization formalism from the corresponding second quantization formalism.

But there is an important difference between the former and the latter formalism. This difference arises in the treatment of the stationary state. In the third quantization formalism there is no clear way to treat that state,

whereas the eigen-states in the second quantization formalism are specified by

$$\mathfrak{H}_{\mathrm{H}}\Psi_s = E_s\Psi_s\,, \tag{47}$$

where $\mathfrak{H}_{\mathrm{H}}$ is the total Hamiltonian of the system.

We can be informed of the properties of $\mathbf{\Omega}^s_{\mathrm{H}}$ from the corresponding Ψ_s by using (37), but if we abandon (37), there is no means to know them. It may be rather undesirable to insist upon the correspondence (37), when we, from the over-all space-time view, want to construct the theory.

Here we suggest a possibility of constructing the theory when we abandon (37).

The first point is that it would be possible to expand the state vectors $\mathbf{\Omega}$ in this space in the form (23), where the base vectors are specified by

$$\begin{cases} \mathfrak{N}_a\mathbf{\Omega}_{nmk} = n\mathbf{\Omega}_{nmk}\,, \\ \mathfrak{N}_b\mathbf{\Omega}_{nmk} = m\mathbf{\Omega}_{nmk}\,, \\ \mathfrak{N}_c\mathbf{\Omega}_{nmk} = k\mathbf{\Omega}_{nmk}\,, \end{cases} \tag{48}$$

with the operators defined as

$$\mathfrak{N}_a = \int \mathrm{d}x\mathfrak{a}^+(x)\mathfrak{a}(x), \qquad \mathfrak{N}_b = \int \mathrm{d}y\mathfrak{b}^+(y)\mathfrak{b}(y)\,, \qquad \mathfrak{N}_c = \int \mathrm{d}z\,\mathfrak{c}^+(z)\,\mathfrak{c}(z)\,,$$

where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are the operators satisfying the commutation relation (28), (29) and (30). But here the expansion coefficients need not be the usual Feynman amplitudes, and of course, if $g(x_\alpha \ldots; y_1 \ldots; z_1 \ldots)$ is taken as the usual Feynman amplitude, we necessarily obtain the correspondence (37).

The second point is that the basic vector $\mathbf{\Omega}_0$ of this space is specified by (32) and the base vectors can be represented by (31). This is necessary for the existence of the minimum in the eigen-value (8), and moreover for the fact that m, n and k are the positive integers.

The third point is to require the conditions (46) in order to specify the state vector $\mathbf{\Omega}_{\mathrm{H}}$. In the previous section we have derived them from (37), but here we assume that they still hold even when we abandon (37).

In order that (46) can be regarded as the substitute of (47) in the second quantization formalism, we have to introduce the following operators:

$$\begin{cases} \bar{\mathfrak{L}} = \int \mathrm{d}^4x\mathfrak{L}(x)\,, \\ \mathfrak{L}(x) = -\dfrac{i}{2}\,\boldsymbol{\varphi}(\Box - \mu^2)\boldsymbol{\varphi} + i\bar{\boldsymbol{\psi}}\left(\gamma\dfrac{\partial}{\partial x} + \varkappa\right)\boldsymbol{\psi} - iL_I(\boldsymbol{\psi}, \bar{\boldsymbol{\psi}}, \boldsymbol{\varphi})\,. \end{cases} \tag{49}$$

Then, (46) can be written as

$$\begin{cases} [\overline{\mathfrak{L}},\ \mathfrak{a}^{+}(x)]\mathbf{\Omega}_{\mathrm{H}} = \mathfrak{a}^{+}(x)\mathbf{\Omega}_{\mathrm{H}}\,, \\ [\overline{\mathfrak{L}},\ \mathfrak{b}^{+}(x)]\mathbf{\Omega}_{\mathrm{H}} = \mathfrak{b}^{+}(x)\mathbf{\Omega}_{\mathrm{H}}\,, \\ [\overline{\mathfrak{L}},\ \mathfrak{c}^{+}(x)]\mathbf{\Omega}_{\mathrm{H}} = \mathfrak{c}^{+}(x)\mathbf{\Omega}_{\mathrm{H}}\,. \end{cases} \tag{50}$$

To keep the analogy with (47), $\mathbf{\Omega}_{\mathrm{H}}$ is required to be an eigen-vector of $\mathfrak{L}$ with eigen-value l:

$$\overline{\mathfrak{L}}\mathbf{\Omega}_{\mathrm{H}} = l\mathbf{\Omega}_{\mathrm{H}}\,. \tag{51}$$

Equation (50) is now reduced to

$$\overline{\mathfrak{L}}(\mathfrak{a}^{+}(x)\mathbf{\Omega}_{\mathrm{H}}) = (l+1)(\mathfrak{a}^{+}(x)\mathbf{\Omega}_{\mathrm{H}}),\ \text{etc.}. \tag{52}$$

Consequently $\mathfrak{a}^{+}\mathbf{\Omega}_{\mathrm{H}}$, $\mathfrak{b}^{+}\mathbf{\Omega}_{\mathrm{H}}$ and $\mathfrak{c}^{+}\mathbf{\Omega}_{\mathrm{H}}$ are found to be the eigenvectors with a common eigenvalue $l+1$. Thus instead of solving (6) it will suffice to solve the eigen-value problem (51) with the subsidiary condition (52). Unfortunately, however, (51) is not the eigen-value problem in the usual sense. This is due to the fact that $\mathfrak{L}$ is not the hermitian operator and consequently l is not a real number.

To overcome this difficulty it would be necessary to modify the requirement (46), so as to hermitize the quantity $\mathfrak{L}$. This attempt should be examined in the future, and it might be necessary to bring entirely new concepts (for instance, such as the negative energy particles).

* * *

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RIASSUNTO (*)

Per riformulare la teoria quantistica del campo dal punto di vista dello spazio- tempo generale, prendiamo le ampiezze di Feynman costruite per mezzo degli operatori di seconda quantizzazione. Trattando queste ampiezze di Feynman, si introducono gli operatori di terza quantizzazione che creano e distruggono gli operatori di seconda quantizzazione. Questi operatori di terza quantizzazione giacciono in un nuovo spazio hilbertiano e operano sui vettori di stato in questo spazio. Si esamina la connessione tra questa formulazione e la consueta teoria della seconda quantizzazione. Si dimostra che questo formalismo è equivalente alla teoria di Coester. Finalmente, servendosi del formalismo della terza quantizzazione, si discute la possibilità di un'ulteriore estensione della descrizione per mezzo dello spazio-tempo generale.

(*) *Traduzione a cura della Redazione.*

Sullo schema di decadimento del $^{214}_{83}$Bi (RaC).

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Riassunto. — Si sono proseguite le ricerche relative allo spettro del $^{214}_{83}$Bi, mediante analisi della curva di assorbimento delle radiazioni β e determinazione delle coincidenze β-γ, con discriminazione integrale dei raggi γ del $^{214}_{84}$Po. I risultati ottenuti si accordano con quelli di Wapstra e di Latishev relativamente all'energia massima della transizione β allo stato fondamentale (3.17 MeV) (intensità relativa 13%) e confermano la presenza di una transizione di energia 2.56 MeV al primo stato eccitato del $^{214}_{84}$Po, da noi antecedentemente segnalata, di intensità relativa pari al 6% circa. L'analisi dello spettro indica inoltre la presenza di altre transizioni β rispettivamente di energia 1.720 MeV (intensità relativa 7%); 1.44 MeV (30%) e 1.03 MeV (22%). Transizioni di energia inferiore a 1 MeV certamente esistono e la loro intensità relativa, secondo i nostri risultati, dovrebbe essere ≈ 20%. Si deduce uno schema di decadimento β del $^{214}_{83}$Bi, il quale è in notevole accordo con lo schema di livelli γ del $^{214}_{84}$Po, recentemente proposto da Demichelis e Malvano.

1. – Introduzione.

Il decadimento radioattivo del $^{214}_{83}$Bi(RaC) si ritiene che avvenga attraverso gli stadi seguenti:

$$^{214}_{83}\mathrm{Bi} \xrightarrow{\beta} {}^{214}_{84}\mathrm{Po}\ (\text{stati eccitati}) \xrightarrow{\gamma} {}^{214}_{84}\mathrm{Po}\ (\text{stato fondamentale}) \xrightarrow{\alpha} {}^{210}_{82}\mathrm{Pb}\ .$$

È stata inoltre accertata dai vari autori che hanno già studiato tale decadimento la presenza della transizione:

$$^{214}_{83}\mathrm{Bi} \xrightarrow{\beta} {}^{214}_{84}\mathrm{Po}\ (\text{stato fondamentale}) \xrightarrow{\alpha} {}^{210}_{82}\mathrm{Pb}\ (^1);$$

(1) G. D. Latishev: *Rev. Mod. Phys.*, **19**, 132 (1947); A. H. Wapstra: *Academisch Proeschrift* (Amsterdam, 1953).

e delle transizioni:

$$^{214}_{83}\mathrm{Bi} \xrightarrow{\beta} {}^{214}_{84}\mathrm{Po}\ (\text{stati eccitati}) \xrightarrow{\alpha(\text{long range})} {}^{210}_{82}\mathrm{Pb}\ (^2)\ .$$

Infine non sono esclusi eventi che corrispondono ad emissioni α di lungo percorso il cui livello energetico di origine non corrisponde a qualcuno dei livelli energetici determinati da transizioni allo stato fondamentale con emissione di raggi γ singoli o in cascata (3).

Quest'ultimo fatto è sufficiente per comprendere come lo schema di decadimento non possa dirsi completamente noto.

Hanno contribuito ad individuare tale schema di decadimento i lavori di Ellis e Aston, di Latishev, di Mladjenovic e Hedgran, per quanto riguarda lo spettro sperimentale dei raggi γ del $^{214}_{84}$Po; i lavori di Latishev e di Wapstra relativamente allo spettro β del $^{214}_{83}$Bi; ed infine i lavori di Surugue, dello stesso Wapstra e di Demichelis e Malvano per quanto riguarda lo schema dei livelli energetici del $^{214}_{84}$Po (4).

Una prima ricerca da noi compiuta ha portato a nuovi fatti per la conoscenza dello spettro di decadimento β del $^{214}_{83}$Bi, già da noi segnalati (5).

Il presente lavoro si riferisce ad una più estesa ricerca su tale schema di decadimento e consta di due parti: la prima riguarda l'analisi completa dello spettro β del $^{214}_{83}$Bi e la determinazione delle intensità relative dei vari rami che lo compongono; la seconda riguarda la determinazione di alcuni degli stati eccitati del $^{214}_{84}$Po, a cui portano le transizioni β provenienti dal $^{214}_{83}$Bi, mediante la tecnica delle coincidenze β-γ.

2. – Preparato radioattivo e dispositivo sperimentale.

La sorgente utilizzata, preparata da due anni, consisteva in una deposizione di $^{226}_{88}$Ra sopra una lamina d'oro, di attività $\approx$ 16 microcurie, sulla quale era stato depositato un sottile strato di vernice in modo che essa fosse in equilibrio radioattivo con i suoi prodotti di decadimento fino al $^{210}_{82}$Pb. Lo strato di vernice e lo spessore d'aria (17 mm) compreso fra la sorgente ed il ricevitore β erano sufficienti ad arrestare completamente i raggi α provenienti dalla sorgente.

(2) E. Rutherford, W. B. Lewis e B. W. Bowden: *Proc. Roy. Soc.*, A **142**, 347 (1933).

(3) F. Demichelis: *Nuovo Cimento*, **12**, 407 (1954).

(4) C. D. Ellis e G. H. Aston: *Proc. Roy. Soc.*, A **129**, 180 (1930); C. D. Ellis: *Proc. Roy. Soc.*, A **143**, 350 (1935); G. D. Latishev (1), M. Mladjenovic e A. Hedgran: *Physica*, **18**, 1243 (1952); A. H. Wapstra (1) e J. Surugue: *Journ. de Phys. et Rad.*, **8**, 7, 145 (1946); F. Demichelis e R. Malvano: *Nuovo Cimento*, **12**, 358 (1954).

(5) R. A. Ricci e G. Trivero: *Rend. Acc. Linc.*, **8**, 17, 44 (1954); *Nuovo Cimento*, **1**, 717 (1955).

Il preparato era quindi da considerarsi come sorgente di radiazioni β e γ provenienti dai diversi nuclidi raccolti nella Tabella I (*).

TABELLA I.

Nuclide	Radiazione	Energia della radiazione
$^{214}_{82}$Pb(RaB)	β	0.65 MeV
$^{214}_{83}$Bi (RaC)	β	comprese tra 0.3 e 3.17 MeV
$^{210}_{82}$Pb(RaD)	β	0.0467 MeV; 0.018 MeV
$^{210}_{83}$Bi (RaE)	β	1.17 MeV (5%)
$^{210}_{81}$Tl (RaC″)	β	1.8 MeV (0.04%)
$^{222}_{86}$Rn	γ	0.184 MeV (1%)
$^{214}_{83}$Bi (RaC)	γ	0.29 MeV; 0.35 MeV
$^{214}_{84}$Po(RaC′)	γ	comprese tra 0.42 e 2.42 MeV
$^{210}_{83}$Bi (RaE)	γ	0.046 MeV

L'influenza delle radiazioni β del $^{210}_{81}$Tl è trascurabile a causa della piccola probabilità della transizione $^{214}_{83}\text{Bi} \xrightarrow{\alpha} {}^{210}_{81}\text{Tl}$ (0.04%); d'altra parte le radiazioni β del $^{210}_{83}$Bi non possono ritenersi presenti che in una piccola percentuale: 5% nel nostro caso (**).

Nel dispositivo di rivelazione β da noi usato (fig. 1), la soglia del formatore di impulsi provenienti dal ricevitore β era tale che gli impulsi dovuti alle radiazioni β provenienti dal $^{210}_{82}$Pb (energia 0.0467 MeV) non venivano rivelate. Le sole radiazioni β rivelabli erano quelle del $^{214}_{82}$Pb e del $^{214}_{83}$Bi, a meno della correzione del 5% dovuta alla presenza di $^{210}_{83}$Bi, non in equilibrio.

Il dispositivo di rivelazione γ (fig. 6), permetteva di segnalare solo impulsi corrispondenti a raggi γ di energia superiore a 0.35 MeV; i soli raggi γ rivelabili erano dunque quelli provenienti dal $^{214}_{84}$Po.

(*) Cfr. *Nuclear Data National Bureau Standards.* Circular 494; vedi anche [4,5,9].

(**) Si può calcolare l'intensità relativa delle disintegrazioni β del $^{210}_{83}$Bi considerando che la sorgente è in equilibrio fino al $^{210}_{82}$Pb e tenendo conto del tempo trascorso dalla sua preparazione (2 anni). Il rapporto tra il numero di atomi di $^{210}_{82}$Pb che si disintegrano al tempo t (formando atomi di $^{210}_{83}$Bi) e il numero di atomi dello stesso $^{210}_{82}$Pb presenti all'equilibrio è, con buona approssimazione: $N_t/N_0 = 1/(1 - e^{-\lambda t})$, dove λ è la costante di decadimento del $^{210}_{82}$Pb. Per $t = 2$ anni, si ha: $N_t/N_0 = 0.05$. N_0 è il numero totale di disintegrazioni al secondo e può assumersi a misura della intensità assoluta della sorgente. Lo stesso N_0 è anche il numero di disintegrazioni al secondo per ciascuno dei nuclidi $^{214}_{82}$Pb e $^{214}_{83}$Bi i quali sono in equilibrio con la sorgente. Poichè l'efficienza ε del contatore β è la stessa per tutte le radiazioni β rivelate, N_t/N_0 dà anche il rapporto tra il numero di impulsi contati relativi al $^{210}_{83}$Bi e il numero di impulsi relativi al $^{214}_{83}$Bi. Per intensità relativa di un ramo (branching ratio) intenderemo sempre il numero di disintegrazioni al secondo di ciascun ramo riferito all'intensità della sorgente.

3. Curva di assorbimento e determinazione delle energie β.

La prima parte della nostra ricerca riguarda la determinazione della curva di assorbimento totale, ossia della curva che, per il nostro preparato e relativamente all'efficienza del dispositivo di conteggio, esprime il numero di eventi al secondo registrati in funzione dello spessore di assorbitore (Al) interposto tra sorgente e contatore β.

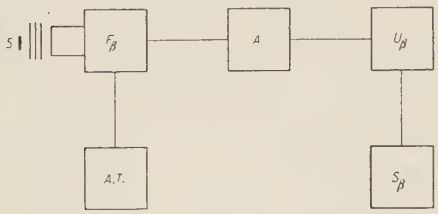

Fig. 1.

In fig. 1 è indicato lo schema del dispositivo usato; in esso S è la sorgente, F_β è un contatore a scintillazione (cristallo di stilbene + fotomoltiplicatore RCA 5819) seguito da un amplificatore lineare A e da un formatore U_β di impulsi, i quali vengono segnalati da un normale circuito di conteggio S_β. Il procedimento sperimentale seguito è analogo a quello relativo alla prima serie di misure già da noi pubblicate [5].

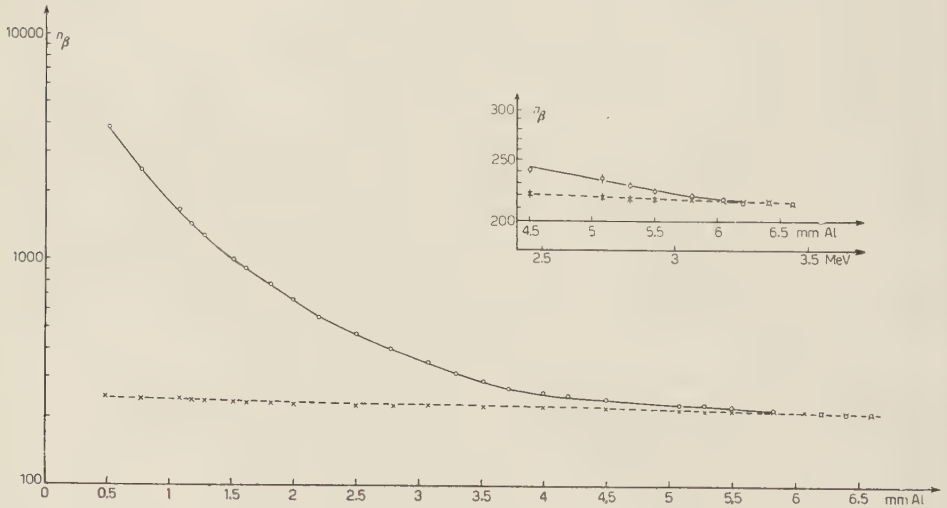

Fig. 2. – (n_β, eventi al secondo).

I risultati sono raccolti nel diagramma di fig. 2, in cui sono riportati, sulle ascisse, gli spessori, in mm, di Al e, sulle ordinate, in scala logaritmica, il numero n_β di impulsi al secondo. La linea tratteggiata corrisponde agli eventi di fondo cioè al numero di eventi al secondo ancora registrati, dopo aver interposto tra la sorgente e il contatore β un ulteriore spessore di Be (9 mm), il quale assorbe completamente i raggi β senza disturbare la rivelazione dei raggi γ, che costituiscono essenzialmente gli eventi di fondo registrati dal contatore β [1,5].

Dal particolare riportato in alto a destra della fig. 2 si vede che il punto di intersezione della curva totale con quella del fondo è ben determinabile ed è compreso fra 5.6 mm e 6.2 mm di Al. L'energia massima corrispondente allo spessore ottenuto per estrapolazione della parte estrema della curva è uguale a 3.14 MeV secondo la relazione di FEATHER ([6]). Nei limiti delle nostre approssimazioni si è assunto il valore (3.14 $\pm$ 0.16) MeV, in buon accordo con il valore trovato da LATISHEV e da WAPSTRA (3.17 MeV). Tale valore corrisponde quindi alla transizione β di massima energia del $^{214}_{83}$Bi.

Si è proceduto quindi alla analisi della curva di assorbimento al fine di determinare le energie dei vari rami che compongono lo spettro complesso del $^{214}_{83}$Bi, e le loro intensità relative.

Tale analisi non è agevole come nel caso delle analisi spettromagnetiche. È noto che gli svantaggi che presenta il metodo dell'assorbimento nella determinazione delle energie degli spettri β sono essenzialmente dovuti alla difficoltà di ottenere una precisa relazione « range-energia », alla minore accuratezza rispetto al metodo spettromagnetico ed al fatto che esso non permette una fedele riproduzione della forma dello spettro. D'altra parte l'analisi per assorbimento presenta notevoli vantaggi quali la semplicità, la rapidità e, soprattutto la sensibilità, per cui molti sono gli autori che usano tale metodo nelle ricerche sperimentali relative agli schemi di decadimento ([1,7,10,11]).

I metodi di analisi da noi seguiti sono due: il metodo di Bleuler e Zünti ([7]) ed il metodo delle n-me potenze ([10]). I risultati ottenuti sono tra di loro in accordo e, come si vedrà, soprattutto con il secondo metodo è possibile compiere una analisi soddisfacente dello spettro β.

4. – Il metodo di Bleuler e Zünti.

In fig. 3 è riportata l'intera curva di assorbimento ottenuta depurando la curva totale di fig. 2 dalla curva del fondo.

Secondo il metodo di Bleuler e Zünti è possibile tracciare la curva di assorbimento di un singolo spettro β conoscendo i valori d_n degli spessori di Al per cui la intensità relativa dello spettro si riduce di un fattore α^n ($n = 1, 2, 3, \ldots$)

([6]) N. FEATHER: *Proc. Cambridge Phil. Soc.*, **34**, 599 (1938).

([7]) E. BLEULER e W. ZÜNTI: *Helv. Phys. Acta*, **19**, 383 (1946).

([8]) S. DEVONS: *Excited states of nuclei* (Cambridge, 1949); C. S. WU: *Rev. Mod. Phys.*, **22**, 327 (1954).

([9]) R. W. KING: *Rev. Mod. Phys.*, **26**, 327 (1954).

([10]) L. KATZ e A. S. PENDOLF: *Rev. Mod. Phys.*, **24**, 28 (1952).

([11]) L. ALVAREZ: *Phys. Rev.*, **75**, 1815 (1949), R. N. HASLAM, L. KATZ, H. J. MOODY e H. M. SKARSGARD: *Phys. Rev.*, **80**, 318 (1950).

determinabile dalla curva stessa e quando se ne sia determinata l'energia massima.

Nel nostro caso è $\alpha = 1.5$: i valori d_n si sono desunti dalle curve semiempiriche di Bleuler e Zünti e si è tenuto conto delle correzioni da apportare per la geometria dell'apparato e per il numero atomico del nuclide radioattivo β.

In fig. 3 sono appunto riportate le 4 curve singole di assorbimento in cui si può ragionevolmente scomporre la curva totale sperimentale, quando per ciascun ramo singolo si applichi il metodo suddetto.

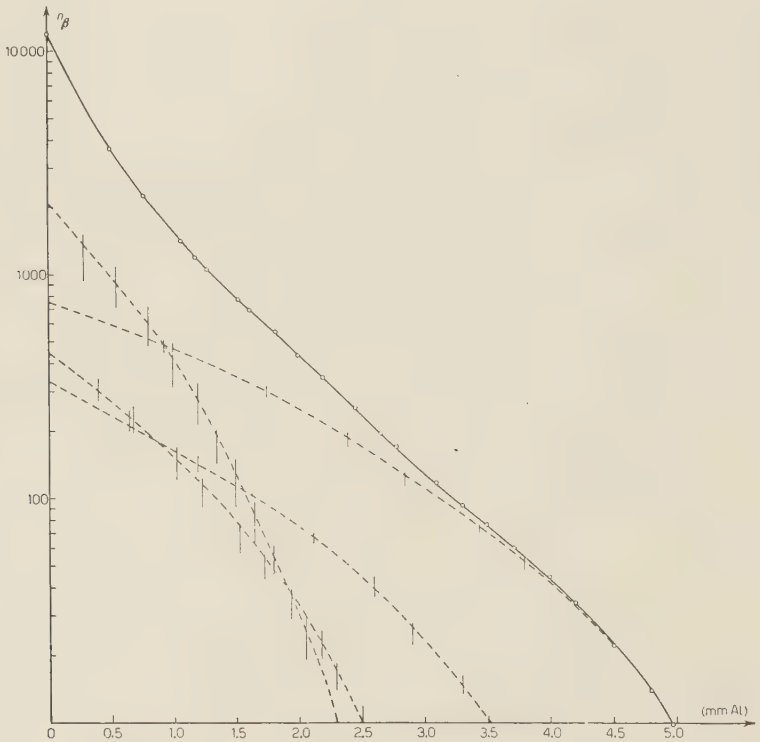

Fig. 3. n_β, eventi al secondo.

Si possono quindi considerare sperimentalmente individuati 4 rami β corrispondenti a 4 spettri singoli di energia massima:

$$(3.14 \pm 0.16)\ \text{MeV}, \qquad (2.47 \pm 0.15)\ \text{MeV},$$

$$(1.71 \pm 0.14)\ \text{MeV}, \qquad (1.40 \pm 0.17)\ \text{MeV}.$$

È necessario d'altra parte osservare che il metodo di Bleuler e Zünti presuppone spettri β di forma permessa, corrispondenti a transizioni β permesse

($\Delta J = 0; 1$; senza cambio di parità) o, al più, proibite del primo ordine ($\Delta J = 0; 1$; cambio di parità); queste ultime, per elevati valori di Z (come è nel nostro caso), assumono forme permesse ([8]).

Secondo WAPSTRA tuttavia, la transizione 3.17 MeV apparterrebbe alla classe speciale delle transizioni proibite del 1° ordine ($\Delta J = 2$, cambio di parità) ed in tal caso lo spettro corrispondente potrebbe non avere forma permessa; secondo i dati della tabella recentemente pubblicata da KING ([9]), tale transizione, essendo una transizione $(1^-) \rightarrow (0^+)$, sarebbe invece da ascriversi tra le transizioni normali del 1° ordine.

D'altro lato i risultati ottenibili col metodo di Bleuler e Zünti sono senz'altro in accordo con quelli da noi trovati con altri metodi descritti qui in seguito e questo farebbe ritenere che la transizione 3.17 MeV debba avere forma permessa.

L'estrapolazione allo spessore zero delle curve in cui si scinde il diagramma di fig. 3 permette di stabilire le intensità relative delle varie transizioni β del $^{214}_{83}$Bi dal computo degli eventi totali registrati, dovuti, come si è visto, al $^{214}_{82}$Pb, al $^{214}_{83}$Bi e al $^{210}_{83}$Bi (5%), e degli eventi corrispondenti ai singoli rami trovati (*). I valori ottenuti sono raccolti nella Tabella II.

TABELLA II.

Nuclide	Energia della transizione	Eventi al secondo	Intensità relativa
$^{214}_{82}$Pb	0.65 MeV	5700	100%
$^{214}_{83}$Bi	(1.40 ± 0.17) MeV	(2040 ± 480)	(35.8 ± 8.5)%
	(1.71 ± 0.14) MeV	(445 ± 77)	(7.8 ± 1.3)%
	(2.47 ± 0.15) MeV	(340 ± 36)	(6.0 ± 0.6)%
	(3.14 ± 0.16) MeV	(750 ± 45)	(13.2 ± 0.8)%
$^{210}_{83}$Bi	1.17 MeV	285	5%

5. – Il metodo delle n-esime potenze.

Un altro metodo che permette una analisi più rigorosa degli spettri β è da ritenersi quello delle n-me potenze introdotto da KATZ e PENFOLD ([10]); esso

(*) Il numero totale εN degli eventi registrati dal contatore β, in assenza di assorbitore era di 11700 al secondo; poichè: $\varepsilon N = \varepsilon(N_{RaB} + N_{RaC} + N_{RaE})$; posto $N_{RaB} = N_{RaC} = N_0$ (intensità della sorgente in equilibrio), ed essendo $N_{RaE} = 0.05 N_0$, si ha $\varepsilon N_0 = 5700$ eventi al secondo. Le intensità relative dei vari rami risultano dal rapporto

$$\frac{\varepsilon N_p}{\varepsilon N_0} = \frac{N_p}{N_0},$$

dove N_p è il numero di eventi parziali al secondo e quindi εN_p è il numero di eventi al secondo contati.

ha dato buoni risultati in tutti i casi in cui finora è stato applicato [10,11].

Secondo tale metodo, da una curva di assorbimento accuratamente determinata, è possibile, per successive approssimazioni, determinare le energie delle transizioni β che compongono uno spettro complesso con una precisione

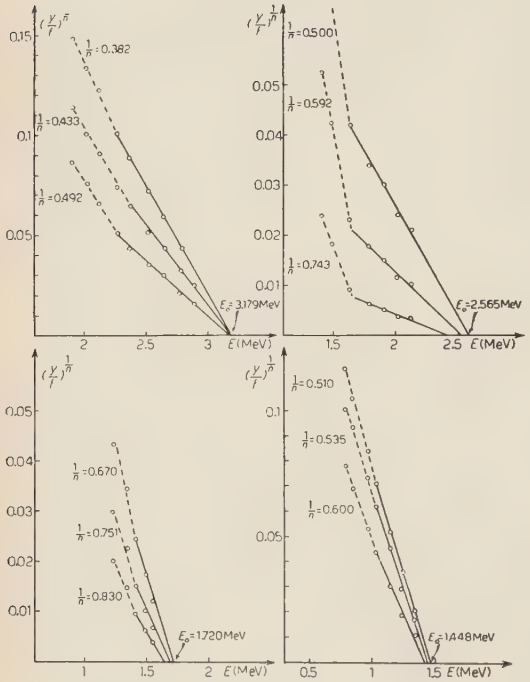

Fig. 4.

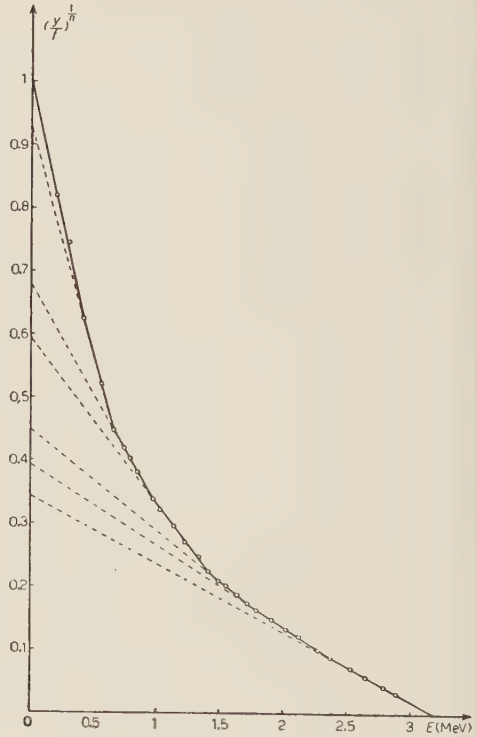

Fig. 5.

maggiore che non con altri metodi analoghi; inoltre è possibile, con opportune correzioni, ottenere una curva simile a quella che fornisce un Kurie-plot, come si ottiene nelle analisi spettromagnetiche. Ciò permette una più agevole determinazione delle intensità relative delle transizioni che si sono potute individuare.

L'applicazione di questo metodo al nostro caso ci ha permesso di compiere una analisi dello spettro β del $^{214}_{83}Bi$ molto più completa di quanto non fosse possibile con il metodo di Bleuler e Zünti. L'approssimazione con cui si sono potute determinare le energie dei rami non supera il 5%, e quella con cui si sono determinate le intensità relative è inferiore al 7.5%.

In fig. 4 sono riportati i diagrammi ottenuti, per approssimazioni successive, relativi ai primi 4 rami individuati nell'analisi dello spettro β.

In ordinate sono riportati i valori $(y/f)^{1/n}$, dove y è la frazione di elettroni ancora trasmessa da un certo spessore di Al ed f un opportuno fattore di cor-

rezione; in ascisse sono riportate le energie massime E corrispondenti ai vari spessori di Al, secondo la relazione di Katz e Penfold per gli spettri β ([10]).

Le energie trovate per i 4 rami β sono rispettivamente: (3.179 ± 0.090) MeV, (2.565 ± 0.125) MeV, (1.720 ± 0.090) MeV, (1.448 ± 0.045) MeV.

In fig. 5 è riportato il diagramma totale $(y/f)^{1/n}$ ottenuto dalla 4° approssimazione del diagramma relativo alla transizione estrema (3.17 MeV).

È possibile osservare la serie di rette corrispondenti ai vari spettri parziali che compongono lo spettro β complessivo. Per estrapolazione a zero di tali rette è possibile determinare le intensità relative dei vari rami.

Dal diagramma di fig. 5 è inoltre possibile determinare un quinto ramo β di energia (1.03 ± 0.08) MeV che, insieme a quelli indicati sopra, determinati per approssimazioni successive, certamente appartiene allo spettro β del $^{214}_{83}$Bi; il ramo β di 0.65 MeV circa, che pure è ben individuabile, è invece da ascriversi al $^{214}_{82}$Pb. Ciò è confermato dal fatto che l'intensità relativa di questo ramo è esattamente il 100%.

Sono certamente presenti transizioni o gruppi di transizioni β di energia intorno a 0.4-0.5 MeV, come può vedersi dal solito diagramma, e la cui intensità relativa complessiva è di circa il 20%. In conclusione il metodo di Katz e Penfold non solo conforta i risultati ottenuti col metodo di Bleuler e Zünti, ma ne permette l'estensione, come è indicato nella Tabella III.

Tabella III.

Energia della transizione	Intensità relativa	Nuclide
(3.179 ± 0.090) MeV	(13.1 ± 0.6)%	$^{214}_{83}$Bi
(2.565 ± 0.125) MeV	(6.0 ± 0.3)%	»
(1.720 ± 0.090) MeV	(6.7 ± 0.5)%	»
(1.448 ± 0.045) MeV	(29.3 ± 2.0)%	»
(1.03 ± 0.12) MeV	(22.2 ± 1.3)%	»
≅ 0.65 MeV	100%	$^{214}_{82}$Pb
≅ 0.45 MeV	≅ 20%	$^{214}_{83}$Bi

6. – Coincidenze β–γ.

All'evento corrispondente ad una emissione β del $^{214}_{83}$Bi e quindi alla creazione del nuclide $^{214}_{84}$Po, succede di regola una emissione γ che avviene in un tempo legato a quello della vita media del corrispondente stato eccitato del $^{214}_{84}$Po creato. Si può ritenere che le vite medie degli stati eccitati di questo nuclide siano dello stesso ordine di grandezza delle vite medie normali degli

stati nucleari eccitati più conosciuti ($\simeq 10^{-12}$ s) e quindi tali da giustificare la considerazione di coincidenze determinabili con un dispositivo a coincidenze avente un tempo di risoluzione dell'ordine di 10^{-7} s.

Il dispositivo da noi usato è schematicamente indicato in fig. 6; in esso F_γ è un contatore a scintillazione (cristallo di NaI(Tl) Harshaw a buon potere risolutore + fotomoltiplicatore Du Mond 6291) usato come contatore proporzionale γ; U_γ è un formatore di impulsi usato come discriminatore integrale; C è un circuito a coincidenze con tempo di risoluzione $(3.35 \pm 0.03) \cdot 10^{-7}$ s.

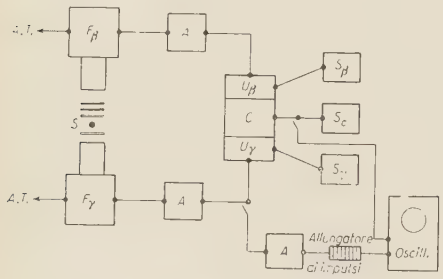

Fig. 6.

Il problema della determinazione delle coincidenze β-γ è reso complesso dal fatto che le radiazioni β non sono monocromatiche, dal gran numero di radiazioni γ provenienti dal $^{211}_{84}$Po e dalla presenza nello spettro di quest'ultimo di numerose cascate γ-γ. Tuttavia lo stesso metodo dell'assorbimento permette una certa discriminazione tra i vari gruppi di raggi β mentre il discriminatore U_γ, la cui tensione di polarizzazione di griglia è stata tarata in funzione dell'energia γ secondo il metodo fotografico-oscillografico descritto da Demichelis e Malvano [3], permette a sua volta di procedere a una certa separazione nel gruppo piuttosto complesso dei γ monocromatici.

La curva delle coincidenze β-γ, determinate facendo pervenire al ricevitore γ l'intero spettro γ del $^{214}_{84}$Po, e riportata in fig. 7, mostra che queste coincidenze cessano per uno spessore di Al di $\simeq 4.5$ mm. Con raggi β appartenenti alla transizione estrema di energia 3.17 MeV non si hanno quindi più coincidenze.

Si deve concludere, a conferma di quanto trovato da Wapstra, che il 13 % dell'emissione β va attribuito ad una transizione di energia 3.17 MeV la quale conduce: o allo stato fondamentale del $^{211}_{84}$Po o ad uno stato eccitato del $^{214}_{84}$Po avente lunga vita media ($> 4 \cdot 10^{-7}$ s). Quest'ultima conclusione appare così poco verosimile da doversi escludere [1].

L'intersezione della curva totale delle coincidenze β-γ con la curva del fondo (coincidenze γ-γ rivelate dal dispositivo β-γ, e determinate sempre ricoprendo la sorgente con l'assorbitore di Be), avviene nell'intervallo di spessori compresi fra 4.4 mm e 4.7 mm di Al.

Estrapolando anche questa volta la parte estrema della curva, si ottiene un valore definito di 4.56 mm di Al, a cui corrisponde una energia β di 2.56 MeV. Si può concludere quindi che esiste una transizione β di energia (2.56 ± 0.25) MeV in buon accordo con quanto trovato mediante l'analisi della curva di assorbimento β. Questa transizione è seguita da una emissione γ.

Poichè nello spettro del $^{214}_{84}$Po è presente una radiazione γ molto intensa di

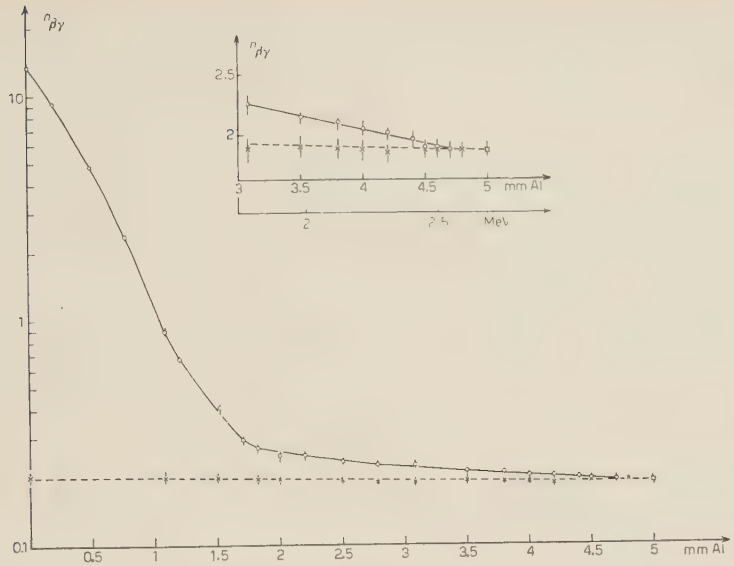

Fig. 7.

energia 0.61 MeV, si può concludere per l'esistenza della transizione β di energia 2.56 MeV, (2.56 + 0.61 = 3.17 MeV), la quale lascia il $^{214}_{84}$Po al 1° stato eccitato.

Se si procede alla discriminazione integrale dei γ emessi dal $^{214}_{84}$Po, si vede che le coincidenze β-γ per spessori di Al compresi tra 3 mm e 4.5 mm dimi-

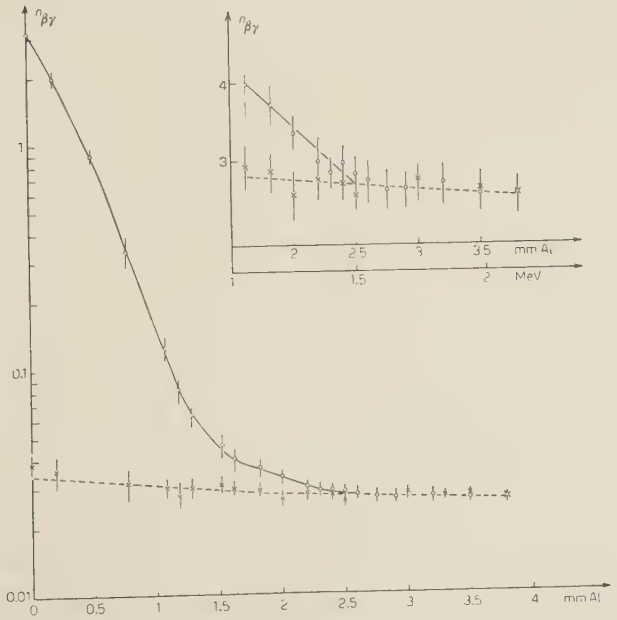

Fig. 8.

nuiscono fino a sparire completamente quando la radiazione γ di 0.61 MeV sia completamente eliminata dallo spettro del $^{214}_{84}$Po (tensione di polarizzazione di griglia del formatore U_γ 17 V).

La fig. 8 rappresenta il diagramma delle coincidenze β-γ ottenute in questo caso.

L'intersezione della curva totale e di quella del fondo avviene, come si vede nel particolare in alto a destra, per uno spessore compreso fra 2.3 e 2.7 mm di Al. La solita estrapolazione indica uno spessore definito di 2.5 mm di Al, a cui corrisponde una energia di 1.49 MeV.

Questo risultato conferma l'esistenza di coincidenze β-γ con la radiazione di 0.61 MeV e quindi la presenza nello spettro β del $^{214}_{83}$Bi della transizione 2.56 MeV.

7. – Discussione dei risultati e conclusioni.

Nella Tabella IV sono riportati per un più comodo confronto, i risultati complessivi delle nostre misure, sia quelli ottenuti per analisi della curva di assorbimento β, sia quelli ottenuti mediante la determinazione di coincidenze β-γ.

Tabella IV.

Energia in MeV			Intensità relativa in %		Nuclide
Analisi col metodo di Bleuler-Zünti	Analisi col metodo delle n-me potenze	Coincidenze β-γ	Metodo di Bleuler-Zünti	Metodo delle n-me potenze	
3.14 ± 0.16	3.179 ± 0.090	—	13.2 ± 0.8	13.1 ± 0.6	$^{214}_{83}$Bi
2.47 ± 0.15	2.565 ± 0.125	2.56 ± 0.25	6.0 ± 0.6	6.0 ± 0.3	»
1.71 ± 0.14	1.720 ± 0.090	—	7.8 ± 1.3	6.7 ± 0.5	»
1.40 ± 0.17	1.448 ± 0.045	1.49 ± 0.14	35.8 ± 8.5	29.3 ± 2.0	»
—	1.03 ± 0.12	—	—	22.2 ± 1.3	»
—	≅ 0.65	—	—	100	$^{214}_{82}$Pb
—	≅ 0.45	—	—	≅ 20	$^{214}_{83}$Bi

In base ai risultati sperimentali si possono fare le seguenti considerazioni e si propongono le seguenti conclusioni:

a) la transizione 3.17 MeV (13%) porta direttamente allo stato fondamentale del $^{214}_{84}$Po in buon accordo con quanto trovato da Wapstra e Latishev [1];

b) si conferma l'esistenza di una transizione 2.56 MeV (intensità relativa 6%), che porta al 1° stato eccitato del $^{214}_{84}$Po [5].

L'intensità relativa complessiva di queste due transizioni è pari alla intensità attribuita da WAPSTRA alla sola transizione 3.17 MeV (19%);

c) si segnala l'esistenza di un ramo β di energia (1.720 ± 0.090) MeV di intensità relativa pari al 7% circa.

È noto che nello spettro γ del $^{214}_{84}$Po esiste il livello 1.414 MeV che non dà luogo a transizioni γ portanti allo stato fondamentale se non attraverso elettroni di conversione interna. Potrebbe quindi assumersi che la transizione β di 1.720 MeV porti a tale stato eccitato e non sia quindi in coincidenza con nessun raggio γ.

Ciò parrebbe confermato dal fatto che allo stato 1.414 MeV deve corrispondere una transizione β di energia 1.756 MeV, valore che non può considerarsi in disaccordo con il valore (1.720 ± 0.090) MeV da noi trovato. Tuttavia si deve osservare che l'intensità relativa di questo livello dovrebbe essere, secondo i dati di Latishev, del 0.34% mentre noi troviamo una intensità relativa del 7% per la transizione β corrispondente; è perciò da escludere che il ramo β da noi trovato sia esaurito dalla transizione a questo stato eccitato.

È quindi ragionevole ritenere che, se esiste un ramo β di energia (1.720 ± ± 0.090) MeV, esso sia costituito in piccola parte dalla transizione 1.756 MeV allo stato 1.414 MeV ed in gran parte da transizioni a livelli γ di energie comprese fra 1.36 MeV e 1.54 MeV.

D'altra parte, come si vede dalla Tabella IV, non vi è accordo fra il valore (1.49 ± 0.14) MeV determinato con le coincidenze β-γ ed il valore (1.720 ± ± 0.090) MeV determinato con l'analisi della curva di assorbimento β, per ciò che riguarda la transizione di energia immediatamente inferiore a 2.56 MeV.

Tuttavia si tenga conto che, quando il discriminatore U_γ è polarizzato in modo da eliminare completamente gli impulsi dovuti ai raggi γ di energia 0.61 MeV, viene diminuita in maniera sensibile la frequenza di conteggio degli impulsi dovuti a raggi γ di energia superiore ([3]), e che la eventuale coincidenza con la transizione di energia 1.720 MeV non appare di elevata intensità relativa (7%).

In tal caso il valore (1.49 ± 0.14) MeV trovato come limite delle coincidenze polarizzate, e delle quali non è determinabile l'intensità col nostro dispositivo, potrebbe attribuirsi o al ramo β di energia (1.720 ± 0.090) MeV, o al ramo β di energia (1.448 ± 0.045) MeV, o infine all'insieme dei due rami.

Ciò non sarebbe quindi in contrasto coll'ipotesi dell'esistenza di livelli γ di energia compresa fra 1.36 e 1.54 MeV. Tuttavia è da osservarsi che, oltre lo stato eccitato 1.414 MeV, non vi sono altri livelli di tali energie, determinati per mezzo delle emissioni α di lungo percorso ([2]).

d) Il ramo di energia (1.448 ± 0.045) MeV di intensità relativa pari al 31% è certamente costituito da due transizioni β, di cui una al livello 1.76 MeV, corrispondente a una transizione γ che porta direttamente allo stato fonda-

mentale (17 %) [4]; e l'altra al livello 1.73 MeV corrispondente sia alla cascata γ-γ: 1.12-0.61 MeV (12 %), determinata da DEMICHELIS e MALVANO [3], sia alla transizione γ di energia 1.73 MeV (2.5 %) recentemente determinata da MLADJENOVIC e HEDGRAN [4],

e) L'energia (1.03 $\pm$ 0.12) MeV va attribuita ad un gruppo di transizioni β agli stati eccitati di energia compresa fra 2.14 e 2 MeV noti per le α di lungo percorso, e corrispondenti a transizioni γ allo stato fondamentale ed ad alcune cascate γ-γ secondo i risultati di DEMICHELIS e MALVANO. L'intensità relativa complessiva di queste transizioni, assumendo come riferimento i dati di LATHISHEV, è di circa il 21 %.

Si nota quindi che vi è buon accordo con il valore del 22 % da noi trovato per il ramo β di energia 1.03 MeV.

f) Esistono certamente transizioni β di energia inferiore ad 1 MeV come è provato dal fatto che esistono stati eccitati di alta energia, conosciuti dalle α di lungo percorso. La loro intensità relativa è, secondo i nostri risultati, complessivamente del 20 % circa.

In fig. 9 è riportato uno schema di decadimento β del $^{211}_{83}$Bi quale risulterebbe dai nostri risultati,

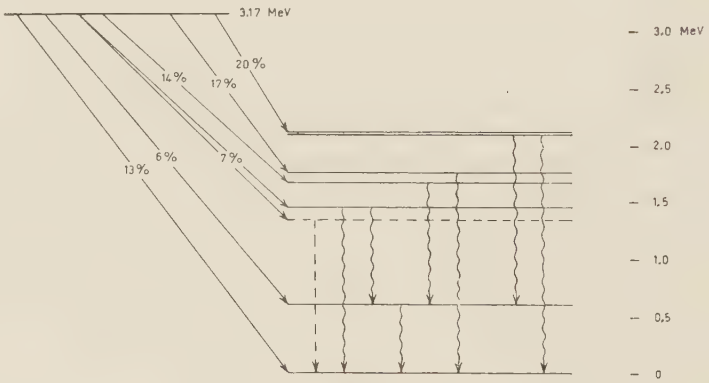

Fig. 9.

Esso è in notevole accordo con lo schema di livelli del $^{214}_{84}$Po recentemente proposto da DEMICHELIS e MALVANO [3], eccetto che per il livello γ, determinato dalla transizione β di 1.72 MeV da noi trovata; tale livello non è considerato nello schema di Demichelis e Malvano.

Si sono infine calcolati, usando i normogrammi di MOSZKOWSKI [12], i valori di log Ft e di log $(W_0^2 - 1)Ft$ per le prime quattro transizioni β da noi determinate.

[12] S. A. MOSZOWSKI: *Phys. Rev.*, **82**, 35 (1951).

Essi sono riportati nella Tabella V insieme col tipo di transizione che se ne può dedurre.

Assumendo, come è noto, che lo stato fondamentale del $^{214}_{84}$Po sia uno stato 0^+ e che il primo stato eccitato sia un 2^+, le nostre conclusioni sono compatibili coll'ipotesi che lo stato fondamentale del $^{214}_{83}$Bi sia uno stato 1^- ([9]), e con l'attribuzione 0^+ allo stato 1.419 MeV di conversione interna ([1]). Da tali conclusioni risulta inoltre che lo stato 1.73 MeV del $^{214}_{84}$Po dovrebbe essere uno stato 2^+.

TABELLA V.

Energia	Intensità	log Ft	log $(W_0^2 - 1)Ft$	ΔJ, cambio parità	Tipo della transizione
3.17 MeV	13%	7.9	9.6	0; 1; 2; si	1° proibita, normale o speciale
2.56 MeV	6%	7.9	9.3	0; 1; si	1° proibita, normale
1.72 MeV	7%	7.15	8.4	0; 1; si	1° proibita, normale
1.44 MeV	14%	6.6	7.7	0; 1; si	1° proibita, normale

D'altra parte, potendosi ascrivere la transizione 3.17 MeV alla classe speciale delle transizioni 1° proibite, i nostri risultati non escludono l'ipotesi di WAPSTRA che lo stato fondamentale del $^{214}_{83}$Bi sia uno stato 2^-.

Teniamo a ringraziare vivamente il Prof. E. PERUCCA, Direttore dell'Istituto, per il costante incoraggiamento, e la Prof. F. DEMICHELIS per la collaborazione nella preparazione sperimentale, le utili discussioni e i suggerimenti. È stato possibile svolgere questa ricerca valendoci anche del contributo finanziario del C.N.R. accordato per le ricerche di Radioattività in corso nel nostro Istituto.

SUMMARY (*)

Experiences have been continued concerning the spectrum of $^{214}_{83}$Bi, analyzing the absorption of the β-radiation and the determination of coincidences β-γ, with complete separation of the γ-rays from $^{214}_{84}$Po. The results are consistent with those of WAPSTRA and LATISHEV referring to the maximum energy of the β-transition in the fundamental state (3.17 MeV) (relative intensity 13%) and confirm the presence of a transition energy of 2.56 MeV at the first excited state of $^{214}_{84}$Po, already reported by us, of relative intensity of about 6%. The spectrum analysis reveals moreover the presence of other β transitions of energy respectively 1.720 MeV (relative intensity 7%); 1.44 MeV (30%) and 1.03 MeV (22%). Transitions of energy less than 1 MeV certainly exist and their relative intensity, following our results, should be ≈ 20%. From this a β-decay scheme may be deduced for $^{214}_{83}$Bi which is in fairly good accord with the scheme of γ-levels of $^{214}_{84}$Po recently proposed by DEMICHELIS and MALVANO.

(*) *Editor's Translation.*

Recherche d'une relation sémi-empirique parcours-énérgie pour les milieux composés.

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(ricevuto il 19 Luglio 1955)

Summary. — A simple analytical relation giving the range of a heavy charged particle (p, α, ...) as a function of its initial kinetic energy is established. This relation contains two parameters, depending on the medium. The ranges calculated are in agreement with the experimental results; the precision is of 1% at the non relativistic energies, which is comparable with the precision given by Vigneron's method. An approximate relation, more rapidly calculable, gives a precision of 2 or 3%. Curves are established in three media: photographic emulsion, air and aluminium.

La formule classique pour la perte d'énergie d'une particule lourde (proton, alpha...) rapide mais non relativiste peut s'écrire

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}R} = -a\frac{Z^2}{M}\frac{\log\left(4m(\mathcal{E}/I)\right)}{\mathcal{E}}, \tag{1}$$

où $\mathcal{E} = E/M$;

M = Masse au repos de la particule (en prenant pour unité la masse du proton);

E = énergie cinétique de la particule (en MeV);

(*) Boursière de l'Institut Interuniversitaire des Sciences Nucléaires.

m = masse au repos de l'électron;

$a = 2\pi e^4 \mathcal{N}/m$;

$\mathcal{N}$ = nombre d'électrons dans un cm³ de matière.

Reprenons la définition donnée par VIGNERON ([1], p. 22) du potentiel moyen d'ionisation I des électrons intervenant dans le ralentissement

$$\log I = \frac{\sum_i N_i Z_i \log I_i}{\sum_i N_i Z_i} ,$$

avec la condition

$$\mathcal{E} > \frac{I_i}{4m} = 0.460 I_i \qquad (I \text{ est exprimé en keV}) .$$

1. – La formule (1) peut s'écrire

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}R} = -a \frac{Z^2}{M} \frac{\log (b\mathcal{E})}{\mathcal{E}} , \tag{2}$$

avec $b = 4m/I = 2.1739/I$; les paramètres a et b ne dépendent donc que du milieu.

Posons $v = 2 \log (b\mathcal{E})$; l'intégration de (2) donne

$$R = \frac{M}{Z^2} \alpha \int_{\underline{v}}^{\bar{v}} \frac{e^v}{v} \mathrm{d}v . \tag{3}$$

Si $\bar{E}$ désigne l'énergie initiale de la particule, la limite supérieure de l'intégrale (3) sera

$$\begin{cases} \bar{v} = 2 \log \bar{\mathcal{E}} + B = 4.6052 \log_{10} \left(\dfrac{\bar{E}}{M} \right) + B , \\ \text{avec} \quad B = -4.6052 \log_{10} (0.460 I) . \end{cases} \tag{4}$$

L'intégration à partir de l'énergie $\underline{\mathcal{E}} = 1/b$ introduit la limite inférieure $\underline{v} = 0$ pour laquelle l'intégrale diverge; pour éviter cette difficulté, nous adopterons $\underline{v} = -\infty$ comme limite inférieure: elle correspond à une intégration sur toutes les énergies de la particule de E à 0. On aura donc

$$\boxed{R = \frac{M}{Z^2} A \int_{-\infty}^{4,6052 \log_{10}(\bar{E}/M) + B} \frac{e^v}{v} \mathrm{d}v} , \tag{5}$$

([1]) L. VIGNERON: *Thèse* (Paris, 1953).

Parcours du $^8Li_3$ dans l'émulsion.

E en MeV	R en μm	$\Delta R/R$	R en μm (5)	$\Delta R/R$	R expér. en μm
4.325	10.13	10	—	—	9.1
5.793	11.6	—	87.22	26	11.6
9.138	18.45	5	16.24	16	19.46
22.41	64.97	—	60.65	6	64.97

Les points expérimentaux son repris chez W. H. BARKAS (2).

Parcours des mésons dans l'émulsion.

Méson	E en MeV	R en μm (5)	$\Delta R/R$	R expér. en μm
π	6	930	3.3	900
π	22.7	10 550	8	9 700
μ	13.6	5 160	6	4 830
μ	28	19 520	20	14 600
μ	33.3	26 300	12	23 205

Les points expérimentaux sont repris chez C. DILWORTH, G. P. S. OCCHIALINI et L. SCARSI (3).

où $A = (m/2\pi e^4 \mathcal{N})(0.4601)^2$ et $B = -4.6052 \log_{10}(0.460 I)$ sont deux paramètres dépendant du milieu.

Le parcours de la particule est donné par la formule (5) sous forme d'une exponentielle intégrale (4)

$$\bar{E}_i(y) = \int_{-\infty}^{y} \frac{e^v}{v}\, dv .$$

VIGNERON (1) a montré qu'il est possible de calculer les paramètres intervenant dans les relations théoriques plutôt que dans les formules empiriques pour obtenir des courbes parcours énergie de grande précision.

2. – Le calcul d'une relation parcours énergie d'après (5) nécessite en principe la connaissance du potentiel I. Souvent celui-ci est inconnu dans les milieux complexes et le problème consiste alors à déterminer, d'après deux points expérimentaux, les paramètres A et B. Ceux-ci ne restent en principe constants que pour des valeurs de l'énergie appartenant à un même intervalle déterminé par les potentiels d'ionisation I_i des atomes du milieu. Les cas où nous avons appliqué la formule montrent que l'on peut encore obtenir une bonne approximation lorsque l'énergie E varie dans plusieurs de ces intervalles.

(2) W. H. BARKAS: *Phys. Rev.*, **89**, 1019 (1953).
(3) C. DILWORTH, G. P. S. OCCHIALINI et L. SCARSI: *A.R.N.S.*, **4** (1954).
(4) JAHNKE et EMDE: *Tables of Functions* (IV édit., 1933), p. 43.

Pour obtenir une courbe parcours-énergie dans un milieu complexe, considérons deux particules

$$P'(E', M', Z') \quad \text{et} \quad P''(E'', M'', Z'') \qquad (E'' > E')$$

Posons

$$(6)\qquad \begin{cases} \bar{x} = 4.6052 \log_{10}\left(\dfrac{E'}{M'}\right) + B\,, \\ \delta = 4.6052 \log_{10}\left(\dfrac{E''}{E'}\cdot\dfrac{M'}{M''}\right), \\ \overline{D} = \log\left(\dfrac{R''}{R'}\cdot\dfrac{Z''}{Z'^2}\cdot\dfrac{M'}{M''}\right). \end{cases}$$

Si les deux particules P' et P'' ont même masse et même charge, les relations (6) se simplifient

$$\delta = 4.6052 \log_{10}\left(\frac{E''}{E'}\right), \qquad \overline{D} = \log\frac{R''}{R'},$$

δ et $\overline{D}$ ne contenant que des grandeurs connues.

A partir de (5) on aura dès lors

$$(7)\qquad \overline{D} = \log_{10}\int_{-\infty}^{\delta+\bar{x}} \frac{e^v}{v}\,dv - \log_{10}\int_{-\infty}^{\bar{x}} \frac{e^v}{v}\,dv\,,$$

Cette relation permet de déterminer $\bar{x}$ et donc B à partir de (6). Une valeur approchée de $\bar{x}$ ne peut être obtenue rapidement à partir de la Fig. 1, où l'on a représenté $D(x)$ pour diverses valeurs du paramètre δ. Si l'on désire obtenir x avec précision (4 ou 5 chiffres) on construira la courbe $D(x)$ au voisinage de $\overline{D}$ (*).

Pour de grandes valeurs de y ($y \geqslant 9$) on peut obtenir une relation parcours énergie approchée plus simple que (5) en utilisant la relation asymptotique

$$(8)\qquad \int_{-\infty}^{y} \frac{v^v}{v}\,dv \cong \frac{1}{y}\,e^y \qquad \text{pour } y \gg 1\,.$$

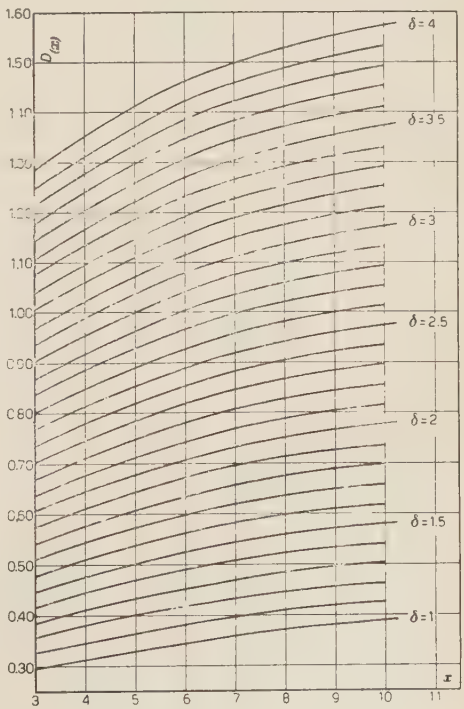

Fig. 1.

(*) Les valeurs numériques de $\log Ei(x)$ construites d'après les « Mathematical

TABLE I.

u	$\log \bar{E}_l(u)$	u	$\log \bar{E}_i(u)$	u	$\log \bar{E}_l(u)$	u	$\log \bar{E}_i(u)$
3.00	0.99712	5.00	1.60406	7.00	2.28218	9.00	3.01614
	0.02925		0.03218		0.03560		0.03772
10	1.02637	10	.63625	10	.31778	10	.05386
	2929		3237		3573		3779
20	.05566	20	.66862	20	.35351	20	.09165
	2931		3256		3586		3786
30	.08497	30	.70118	30	.38937	30	.12951
	2938		3277		3599		3794
40	.11435	40	.71754	40	.42536	40	.16747
	2946		3295		3612		3801
50	.14381	50	.73395	50	.46148	50	.20548
	2956		3314		3624		3808
60	.17337	60	.76690	60	.49772	60	.24357
	2669		3333		3636		3816
70	.20306	5.70	.83337	70	.53408	70	.28173
	2981		3351		3647		3822
80	.23287	80	.86688	80	.57055	80	.31996
	2996		3369		3658		3828
90	.26283	90	.90057	90	.60713	90	.35824
	3011		3388		3670		3835
4.00	1.29294	6.00	1.93444	8.00	2.64383	10.00	3.39659
	0.03028		0.03404		0.03680		0.03841
10	.32322	10	.96849	10	.68063	10	.43500
	3045		3422		3690		3848
20	.35367	20	2.00271	20	.71753	20	.47348
	3064		3437		3700		3855
30	.38431	30	.03708	30	.75453	30	.51202
	3081		3456		3710		3859
40	.41512	40	.07164	8.40	.79163	40	.55061
	3101		3471		3720		3864
50	.44613	50	.10635	50	.82883	50	.58925
	3129		3486		3729		3870
60	.47732	60	.14121	60	.86612	60	.62795
	3138		3503		3738		3876
70	.50871	70	.17624	70	.90350	70	.66671
	3159		3516		3746		3881
80	.54030	80	.21140	80	.94096	80	.70552
	3178		3532		3755		3886
90	.57208	90	.24672	90	.97851	90	.74438
	3198		3546		3763		3891

TABLE I: (*continuation*)

u	$\log \bar{E}_1(u)$	u	$\log \bar{E}_1(u)$	u	$\log \bar{E}_1(u)$	u	$\log \bar{E}_1(u)$
11.00	3.78329	12.00	4.17491	13.00	4.57051	14.00	4.96938
	0.03896		0.03939		0.03975		0.04005
10	.82225	10	.21430	10	.61026	10	5.00943
	3901		3944		3979		4007
20	.86126	20	.25374	20	.65005	20	.04950
	3906		3948		3981		4010
30	.90031	30	.29322	30	.68986	30	.08960
	3910		3951		3984		4012
40	.93941	40	.33273	40	.72970	40	.12972
	3914		3954		3987		4014
50	.97855	50	.37227	50	.76957	50	.16886
	3919		3958		3990		4017
60	4.01774	60	.41185	13.60	.80948	60	.21004
	3924		3962		3993		4020
70	.05698	70	.45147	70	.84941	70	.25024
	3925		3965		3996		4023
80	.09625	80	.49112	80	.88937	80	.29046
	3931		3968		3999		4025
90	.13556	90	.53080	90	.92936	90	.33071
	3935		3971		4002		4026
						15.00	.36827

L'approximation (8) conduit à l'expression

$$R = A' \cdot \frac{M}{Z^2} \cdot \frac{(E/M)^2}{B' + \log_{10}(E/M)} , \tag{9}$$

où A' et B' sont les paramètres à déterminer par deux mesures. Pour B' on

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Tables of the British Association for the Advancement of Science » (5) sont données dans la Table I. Pour des calculs plus détaillés, on se référera aux tables mentionnées (6) dans la bibliographie.

(5) BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE: *Mathematical Tables*, **1**, 31 (1946).

(6) *Tables of Sine, Cosine and Exponential Integrals* (NBS, Washington, 1940).

trouve (lorsque les deux particules ont même masse et même charge)

$$B' = -\frac{(R''/E''^2)\log_{10}(E''/M) - (R'/E'^2)\log_{10}(E'/M)}{(R''/E''^2) - (R'/E'^2)} \quad (10)$$

Cette formule asymptotique donne une bonne approximation dès que la limite supérieure de l'intégrale (5), à savoir

$$4.6052 \log_{10}\left(\frac{E}{M}\right) + B ,$$

est grande (≥ 9). Cette condition est réalisée pour de très grandes énergies ou de très grandes valeurs de B (I petit, éléments légers): cependant, les potentiels d'ionisation I calculés d'après (10) accusent un net désaccord par rapport aux valeurs expérimentales. Ceci est dû au fait que B' est trop sensible aux valeurs numériques utilisées. En effet, lorsque E est grand, R est proportionnel à E^2 et le dénominateur de (10) est très petit. Cette sensibilité ne se présente plus dans le cas de la formule (5) où B donne des valeurs de I en excellent accord avec l'expérience, comme nous le verrons plus loin. La formule (9) se simplifie encore lorsque $\log E/M \ll B'$ et prend alors la forme

$$R = \left(\frac{A'}{MZ^2}\right) E^2 , \quad (11)$$

qui est du type classique $R = AE^m$, formulée par Bohr ([7]).

3. – Applications.

a) Relation R–E dans l'emulsion photographique (cf. tableau d'ensemble I). – Pour les protons, nous avons choisi trois domaines d'énergie, avec les points d'étalonnage suivants:

a)	$E'_p =$	1 MeV	$R' =$ 14.1	μm	([1], p. 39)
	$E''_p =$	7 MeV	$R'' =$ 299	μm	
b)	$E'_p =$	10 MeV	$R' =$ 547.5	μm	([1], p. 39)
	$E''_p =$	39.5 MeV	$R'' =$ 6.123	μm	
c)	$E'_p =$	39.5 MeV	$R' =$ 6.123	mm	([1], p. 39)
	$E''_p =$	342.5 MeV	$R'' =$ 243.5	mm	([8])

([7]) N. Bohr: *Penetration of Atomic Particles through Matter* (1948), p. 126.
([8]) O. Heinz: *Phys. Rev.*, **95**, 1728 (1954).

En déterminant dans ces trois domaines d'énergie les paramètres A', B' de la relation approchée (9) on obtient

$$(12)\qquad \begin{cases} a) & R = \dfrac{9.139E^2}{0.6482 + \log_{10} E}\ ; \\ b) & R = \dfrac{7.669E^2}{0.40208 + \log_{10} E}\ ; \\ c) & R = \dfrac{4.13362}{-0.54328 + \log_{10} E}\ ; \end{cases}$$

La formule exacte (5) n'a été étalonnée que dans le domaine des énergies moyennes. On trouve:

$$\bar{D} = 1.04860\ , \qquad\qquad \delta = 2.7462$$

et ensuite, à partir du graphique I, $\bar{x} = 8.4$. La table I (appendice) permet ensuite de trouver:

$\bar{x} = 8.40$	on trouve	$D(x) = 1.04919$
8.30	»	1.04731
8.35	»	1.04870

et, par interpolation graphique,

$$(13)\qquad \bar{x} = 8.3460 \qquad\qquad B = 3.7408\ .$$

On obtient finalement

$$(14)\qquad R = 0.92637\bar{E}_i[4.6052 \log_{10} E + 3.7408]\ .$$

La valeur de B, donnée par (13) conduit à $I = 335$ eV.

La vérification des formules (12) et (14) pour des protons d'énergie supérieure à 40 MeV. Remarquons que lorsque l'énergie cinétique de la particule atteint le tiers de sa masse au repos, les corrections relativistes ne sont plus négligeables et il faut ajouter au second membre de (1) des termes de la forme

$$a\beta^4 + b\beta^6 + \ldots . \qquad\qquad \text{avec } a, b < 0.$$

Il en résulte que la perte d'énergie exprimée par la relation (1) est inférieure à la valeur réelle et les parcours déterminés sont trop grands.

Parcours des protons dans l'émulsion.

E (MeV)	R en μm d'après les formules du type (11)						R en μm d'après (5)		R en μm d'après VIGNERON		R en μm expér.
	R (11a)	$\Delta R/R$	R (11b)	$\Delta R/R$	R (11c)	$\Delta R/R$	R	$\Delta R/R$	R	$\Delta R/R$	
1	—	—	19.7	35%	—	—	15.2	6 %	14.4	4 %	14.2
2	38.5	3 %	43.6	10	—	—	40.9	2.5	29.23	1	39.7
4	117	1.4	122.2	2.2	—	—	119.6	0.8	118.3	0.2	118.6
16	1 163	7	1222	2.5	1601	22	1230	2	1242	1	1254
25.6	2 913	3	2776	2.8	3132	9	2820	1	2813	1	2849
33.5	4 180	10	4466	3	4725	3	4553	1	4530	1.5	4597
208	—	—	122.0 mm	14	100.8 mm	6	126.5 mm	18	104 mm	3	107.3 mm
342.5	—	—	306.3 mm	26	243.5 mm	—	316.3 mm	23	—	—	243.5 mm

Les parcours des protons de 208 MeV et 342,5 MeV proviennent respectivement des travaux de H. G. DE CARVALHO *et alt.*, (9), et O. HEINZ (8); les autres points sont donnés par L. VIGNERON (1).

$\Delta R/R$ représente l'écart relatif par rapport à la valeur expérimentale.

Parcours des particules α dans l'émulsion.

E en MeV	R en μm (15a)	$\Delta R/R$	R en μm (5)	$\Delta R/R$	R expér. en μm	Références
2.04	6.7	—	6.11	10	6.7	FARAGGI (10)
3.2	10.95	4	11.26	1.3	11.4	»
5.3	21.84	0.7	22.57	2.5	22	»
8.78	47.06	—	47.20	0.3	47.06	LEES *et al.* (11)
11.32	70.8		69.62	1.7	70.8	F.L.C. (12)

(9) H. G. DE CARVALHO et J. I. FRIEDMAN: *Rev. Sci. Instr.*, **26**, 261 (1955).

(10) H. FARAGGI: *Ann. de Phys.*, **6**, 325 (1951).

(11) C. F. LEES, G. C. MORRISON et W. G. V. ROSSER: *Proc. Phys. Soc.*, **66**, 13 (1953).

(12) C. M. G. FOWLER, P. H. LATTES et P. CÜER: *Proc. Phys. Soc.*, **59**, 883 (1947).

Pour les particules α, on peut soit utiliser la relation (9) appliquée aux protons en remplaçant E par E/M, soit procéder à un étalonnage direct. En utilisant les repères suivants:

$$E = 2.04 \text{ MeV} \qquad R = 6.7 \ \mu\text{m} \qquad (^{10}),$$

$$E = 11.32 \text{ MeV} \qquad R = 70.8 \ \mu\text{m}. \qquad (^{12})$$

On obtient la relation

$$R = \frac{0.6233 E^2}{0.07753 + \log_{10} E}. \tag{15a}$$

En partant de la relation (14), on trouve ($M = 3.971$, $Z = 2$)

$$R = 0.91989 \bar{E}_i [4.6052 \log_{10} E_\alpha + 0.9826]. \tag{15b}$$

Les courbes relatives aux particules α et aux protons sont représentées par la fig. 2 et l'accord avec les résultats expérimentaux est satisfaisant.

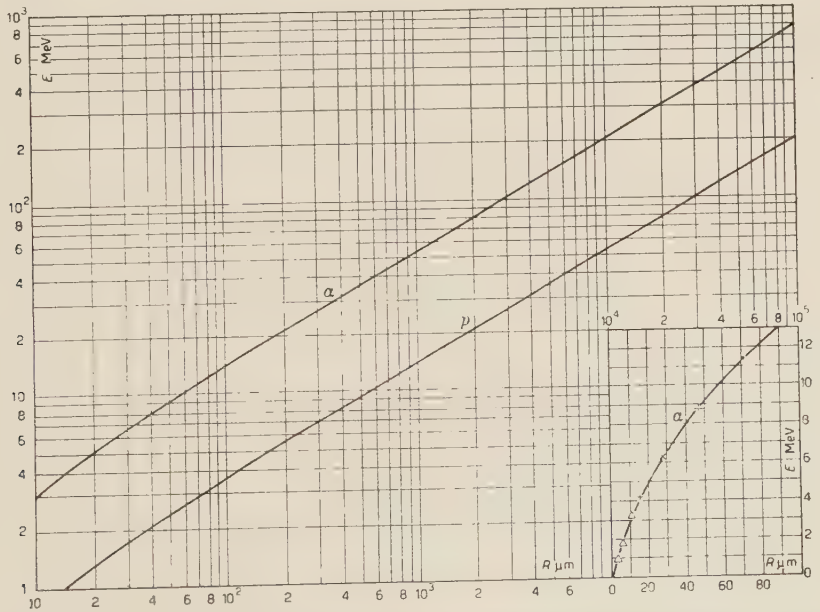

Fig. 2. – Courbe parcours-énergie des protons et des particules α dans l'émulsion photographique. Les points expérimentaux sont repris à P. Cüer et J. P. Lonchamp × ([2]), H. Faraggi △ ([10]), L. Vigneron + ([13]), C. M. G. Fowler, P. H. Lattes et P. Cüer ○ ([12]).

([13]) L. Vigneron: *Compt. Rend. Acad. Sci.*, **237**, 1675 (1953).

Nous avons également vérifié la relation approchée (9) pour des particules de charge plus élevée que 2, en particulier pour le $^8Li_3$. Les repères choisis sont:

$$E = 5.793 \text{ MeV} \qquad R = 11.6 \ \mu\text{m} \quad (^{14})$$

$$E = 22.41 \text{ MeV} \qquad R = 64.9 \ \mu\text{m} .$$

La relation (9) s'écrit

$$R_{^8Li} = \frac{0.1213 E^2}{\log_{10} E - 0.41209} , \tag{16a}$$

et, à partir de (14)

$$R = 0.82344 \overline{E}_i [4.6052 \log_{10} E_{Li} - 0.4181] . \tag{16b}$$

Ces formules ne tiennent pas compte de la possibilité de capture d'électrons. On peut donc s'attendre à ce que les parcours calculés d'après (16*b*) soient inférieurs aux valeurs expérimentales, surtout aux faibles énergies.

Nous avons encore déterminé les relations parcours-énergie pour les mésons π ($M = 0.1504$) et μ ($M = 0.1149$); toujours à partir de (14).

$$R \text{ (mésons } \mu) = 0.10644 \overline{E}_i [4.6052 \log_{10} E + 8.06821] ,$$

$$R \text{ (mésons } \pi) = 0.13933 \overline{E}_i [4.6052 \log_{10} E + 7.52973] .$$

A cause de l'approximation non relativiste, les valeurs sont indiquées au Tableau I et sont effectivement légèrement supérieures aux valeurs expérimentales signalées dans (12). La valeur trouvée pour le méson μ de 28 MeV est en désaccord flagrant par rapport aux autres valeurs.

Remarquons également que les valeurs expérimentales choisies pour l'étalonnage des formules (5) et (11) sont entachées d'une certaine erreur, ce qui entraine une erreur dans la détermination des constantes A et B et limite donc la validité des interpolations et surtout des extrapolations.

b) *Relation parcours-énergie pour les protons dans l'aluminium* — En utilisant les repères suivants:

$$E_1 = 75.84 \text{ MeV} \qquad R = 6.063 \text{ g cm}^{-2} \quad (^9)$$

$$E_2 = 34.96 \text{ MeV} \qquad R = 1.541 \text{ g cm}^{-2} .$$

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(14) A. E. Taylor: *Rep. Progr. Phys.*, **15**, 49 (1952).

on obtient

$$R_{\text{gcm}^{-2}} = \frac{2.163E^2}{0.1719 + \log_{10} E} \qquad \text{(voir fig. 3).}$$

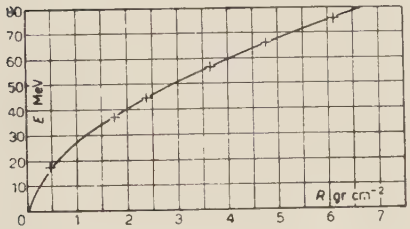

Fig. 3. – Courbe parcours-énergie des particules α dans l'aluminium; les points expérimentaux sont ceux de N. Bloemberger et P. J. Van Heerden [15].

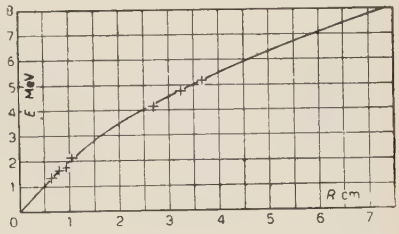

Fig. 4. – Courbe parcours-énergie des particules α dans l'air; les points expérimentaux proviennent de Taylor [14].

c) *Relation parcours-énergie pour les* α *dans l'air.* – En utilisant les repères suivants:

$$E_1 = 1.113 \text{ MeV} \qquad R_2 = 0{,}559 \text{ cm} \quad [16]$$

$$E_2 = 5.51 \text{ MeV} \qquad R_2 = 4.08 \text{ cm}$$

on obtient

$$R_{\text{cm}} = \frac{0.1329E^2}{0.2481 + \log_{10} E} \qquad \text{(voir fig. 4).}$$

4. – Conclusions.

La comparaison des résultats résumés dans le Tableau I nous montre que la relation analytique que nous avons établie donne des écarts relatifs de l'ordre du % par rapport aux données expérimentales, lorsque les énergies sont comprises entre les énergies d'étalonnage.

En dehors de ce domaine, les écarts s'accentuent. Pour les basses énergies, cet écart est principalement dû à la variation des paramètres A, B, variations dues au fait que certains électrons ne participent plus au ralentissement. Cet

[15] N. Bloemberger et P. J. Van Heerden: *Phys. Rev.*, **83**, 561 (1951).
[16] P. Cüer et J. P. Lonchamp: *Compt. Rend. Acad. Sci.*, **232**, 1824 (1951).

écart peut être réduit en choisissant de nouveaux points d'étalonnage. L'écart est plus grand lorsque l'on extrapole vers les hautes énergies: dans ce domaine, il faut tenir compte des effets relativistes et la formule (1) doit être modifiée; l'intégration de la relation (2) ne peut plus s'effectuer simplement.

* * *

Nous tenons à remercier vivement le Professeur Occhialini, le Dr. Franzinetti et le Dr. Vanderhaeghe pour les suggestions qu'ils nous ont faites à l'occasion du présent travail.

RIASSUNTO (*)

Si stabilisce una semplice relazione analitica per il raggio d'azione di una particella pesante carica (p, α ...) in funzione della sua energia cinetica iniziale. Questa relazione contiene due parametri dipendenti dal mezzo. I raggi d'azione calcolati sono in accordo coi risultati sperimentali; la precisione è dell'1% alle energie non relativistiche, il che è confrontabile con la precisione offerta dal metodo di Vigneron. Una relazione approssimata, che si presta a calcoli più rapidi, dà una precisione del 2 o 3%. Le curve sono tracciate per tre mezzi: emulsione fotografica, aria, alluminio.

(*) *Traduzione a cura della Redazione.*

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On the Description of Collective Motion by the Use of Superfluous Co-Ordinates.

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(ricevuto il 19 Luglio 1955)

Summary. — By introducing superfluous co-ordinates a method is demonstrated for treating the problem of combined collective and individual particle aspects of nuclear structure. The application to the case of the center of mass gives a simple exposition of the method and rigorously defines conditions for the validity of the usual shell-model assumptions regarding the center of mass. The application to rotation gives results similar to those obtained by A. Bohr using a hydrodynamical model and by Inglis using a classical rotating shell model. The present treatment is rigorous except for the fundamental shell-model assumption, the replacement of the inter-particle potential by a single-particle potential which may not be spherically symmetric.

1. – Introduction.

The recent success of a nuclear model combining collective and individual particle aspects [1] indicates the desirability of using collective co-ordinates to describe a nuclear system, while at the same time retaining the individual particle co-ordinates. To obtain a set of independent co-ordinates the number of individual particle co-ordinates must be reduced by the number of collective co-ordinates introduced. This leaves the number of degrees of freedom of the system unchanged, but destroys the inherent symmetry of the particle co-ordinates as expressed in the shell model. It is therefore of interest to consider the possibility of retaining *all* the individual particle co-ordinates, in addition

[1] A. Bohr and B. R. Mottelson: *Dan. Mat. Fys. Medd.*, **27**, no. 16 (1953).

to the collective co-ordinates. These co-ordinates are not all independent, and superfluous degrees of freedom are introduced by treating them as if they were. Yet, this is the usual practice in handling the problem of the center of mass in the shell model [2], and it can be extended as well to other modes of collective motion.

A justification for this procedure and a general formalism for the treatment of collective motion in a rigorous manner can be obtained by considering the general question of the introduction of superfluous co-ordinates and degrees of freedom into a quantum mechanical problem.

2. – Quantum Mechanics with Superfluous Co-ordinates.

Consider a dynamical system consisting of A particles having co-ordinates x_i, and for which a set of operators is defined corresponding to all physical observables of the system. These operators are functions of the co-ordinates x_i and their canonically conjugate momenta $p_i = (\hbar/i)(\partial/\partial x_i)$. Let us now introduce a set of extra co-ordinates y_j, having no physical significance. Wave functions can now be written formally as functions $\psi(x_i, y_j)$ of both the original co-ordinates x_i and the superfluous co-ordinates y_j. Furthermore, new operators can be defined which are functions of the y_j and their canonically conjugate momenta $(\hbar/i)(\partial/\partial y_j)$. The new system of co-ordinates (x_i, y_j) possesses additional degrees of freedom having no physical significance in the original problem. Spurious solutions and new quantum numbers without physical significance are likely to appear in the solution of the quantum mechanical problem in the extended co-ordinate system.

2·1. *Physical operators.* – Let us define the term *physical operator* to mean an operator which is a function only of the physical co-ordinates x_i, and their conjugate momenta p_i, and is completely independent of the superfluous co-ordinates y_j and their conjugate momenta. It is evident that a complete set of eigenfunctions of a physical operator can be found having the form:

$$\psi_{mn}(x_i, y_j) = h_m(y_j)\varphi_n(x_i) \,. \tag{2.1}$$

where the functions $h_m(y_j)$ are an arbitrary complete set of normalized functions of the superfluous co-ordinates y_j and $\varphi_n(x_i)$ are just the ordinary complete set of eigenfunctions of the operator in the original physical system before the superfluous co-ordinates were introduced. The eigenvalues of the physical operator are exactly the same as they were in the original system, depending only on the function $\varphi_n(x_i)$ and are completely independent of the form of

[2] A. BOHR: *Rotational States of Atomic Nuclei* (Copenhagen, 1954).

the functions $h_m(y_j)$. Thus if attention is restricted to eigenfunctions and eigenvalues of *physical operators*, the effect of the introduction of the superfluous co-ordinates is only to introduce a spurious degeneracy described by the superfluous quantum number m having no physical significance. Wave functions which are different in the (x_i, y_j) system but which differ only in the quantum number m correspond to the *same* physical wave function in the (x_i) system.

By using a Hamiltonian which is a physical operator, it is possible to set up the Schrödinger equation in the (x_i, y_j) system, to solve it, and to know that to any solution thereby obtained, one can find a solution to the problem in the (x_i) system, with the same eigenvalue. It is also possible to use the eigenfunctions in the (x_i, y_j) system to calculate expectation values of physical operators. This follows from the fact that any complete set of simultaneous eigenfunctions of a set of commuting physical operators must have the product form (2.1) if the set of physical operators is complete in the (x_i) system. Then if B is a physical operator,

$$\frac{\int h^*(y_j)\varphi^*(x_i)Bh(y_j)\varphi(x_i)\,\mathrm{d}x_i\,\mathrm{d}y_j}{\int h^*(y_j)q^*(x_i)\;\; h(y_j)q(x_i)\,\mathrm{d}x_i\,\mathrm{d}y_j} = \frac{\int \varphi^*(x_i)B\varphi(x_i)\,\mathrm{d}x_i}{\int q^*(x_i)\;\; q(x_i)\,\mathrm{d}x_i}\,. \tag{2.2}$$

Now that the physically significant aspects of the problem in the (x_i, y_j) system have been defined, a point transformation can be made to new co-ordinates which are functions of both the x_i and the y_j, so that it is no longer possible to distinguish by inspection between real and superfluous co-ordinates. However, physical operators in the (x_i, y_j) system are transformed into operators in the new system which still have the properties described above; hence they remain physical operators. The physical aspects of the problem in the new system are thus also defined, and calculations with physical operators and their eigenfunctions can be made freely, remembering only that a superfluous degeneracy has been introduced into the system.

2·2. *Semi-physical operators.* – In cases where a physical operator does not have a convenient form, it is possible to introduce a more general set of operators, which we shall call *semi-physical operators*, defined in the following manner:

Let B be a physical operator. Then an operator B' is defined as a *semi-physical operator* corresponding to the physical operator B if it satisfies the relation:

$$[\prod_j \delta(y_j)]B'\varphi(x_i, y_j) = [\prod_j \delta(y_j)]B\varphi(x_i, y_j)\,, \tag{2.3}$$

for all functions $\varphi(x_i, y_j)$ which are finite and regular for $y_j = 0$.

It is evident that if $\varphi(x_i, y_j)$ is an eigenfunction of a semi-physical oper-

ator B' with eigenvalue b, then

$$\psi_p(x_i, y_j) = [\prod_j \delta(y_j)]^{\frac{1}{2}} \varphi(x_i, y_j) , \tag{2.4}$$

is an eigenfunction of the corresponding physical operator B having the same eigenvalue b. Thus semi-physical operators can sometimes be used in place of the corresponding physical operators. The wave functions having the form (2.4) can be called physical wave functions corresponding to the subsidiary condition:

$$y_j \psi_p(x_i, y_j) = 0 \quad ([3]) . \tag{2.5}$$

The square root of the δ function is introduced to give proper normalization. The use of physical wave functions with semi-physical operators is equivalent to the procedure of using superfluous co-ordinates and removing the extraneous degrees of freedom by the use of subsidiary conditions, which in this case are functions only of the superfluous co-ordinates and not of the conjugate momenta. Functions satisfying the subsidiary conditions must therefore contain δ functions.

Since the eigenfunction $\varphi(x_i, y_j)$ of the semi-physical operator is multiplied by δ functions in order to obtain the eigenfunction of the physical operator, it is clear that the dependence of $\varphi(x_i, y_j)$ upon the variables y_j has no physical significance. All that matters is the value of $\varphi(x_i, y_j)$ when all the y_j are set equal to zero. It is therefore possible to write the same physical wave function $\psi_p(x_i, y_j)$ in an infinite variety of ways, using different functions $\varphi(x_i, y_j)$ which need be equal to one another only when all the y_j are set equal to zero. Thus the superfluous degeneracy introduced into the system by the additional co-ordinate is eliminated only after the function $\varphi(x_i, y_j)$ is multiplied by the δ function. This degeneracy still exists in the eigenfunctions of semi-physical operators.

In this work the spurious degeneracy is not considered to be troublesome, and no attempt is made to remove it. VILLARS (*) has considered this question and removes the degeneracy by imposing a subsidiary condition upon φ, namely that it be independent of y_j.

Semi-physical operators may introduce extraneous eigenvalues corresponding to physical wave functions which are identically zero. These pathological

([3]) In order to avoid any difficulties connected with the square root of the δ function, equation (2.5) can be interpreted to mean that the matrix element of the operator y_j vanishes between all pairs of physical wave functions.

(*) F. VILLARS: Private communication. Dr. VILLARS has independently treated the same problem discussed in this paper by essentially the same methods. The authors are indebted to Dr. VILLARS for sending them a copy of his manuscript and for several interesting comments.

cases can generally be recognized by inspection and rejected. A simple example of this is the semi-physical operator $B' = B + y_j$. The eigenfunctions of this operator have the form $\delta(y_j - c)\varphi_b(x_i)$, where $\varphi_b(x_i)$ is the eigenfunction of the physical operator B corresponding to the eigenvalue b, and c is any arbitrary constant. The eigenvalue of the operator B' is $b + c$ and all functions for which $c \neq 0$ are extraneous and correspond to physical wave functions which are identically zero (4).

2·3. *Interpretation of results obtained by use of superfluous co-ordinates.* – By means of the introduction of superfluous co-ordinates into a given physical problem, a new problem is constructed with additional superfluous degrees of freedom. It has been shown in the preceding sections that the following results in the extended system are applicable directly to the original physical problem:

a) All eigenvalues of physical operators are eigenvalues of the corresponding operator in the original system; the corresponding eigenfunction in the original system can be constructed if the eigenfunction in the extended system is known.

b) All expectation values of physical operators calculated for states of the type (2.1) which are simultaneous eigenstates of a complete commuting set of physical operators apply to the original system as well.

c) The property (*a*) applies to semi-physical operators as well, except for certain pathological cases where the physical wave function vanishes.

It is often convenient to use approximate methods in the solution of this type of problem, and these may lead to the use of operators which are neither physical nor semi-physical. This is perfectly legitimate, as long as the only results which are carried over to the actual physical system are results of the type (*a* - *c*) above.

The exact calculation of results of type (*a*-*c*) in the extended system is a well defined mathematical problem. Therefore any procedure for obtaining approximate results is legitimate if the error in the result can be shown to be small. There is no necessity for the intermediate stages in the calculation to have any physical significance. It is therefore possible to treat the problem in the extended system by the usual approximate methods of quantum mechanics, provided that only results of type (*a*-*c*) are given physical significance. It is permissible, for example, to split a semi-physical operator into two parts, a principal part and a perturbation, neither of which is physical or semi-physical, as long as all that is desired are eigenvalues. The sole criterion for

(4) The writers are indebted to Prof. G. Racah for directing their attention to these pathological cases.

the validity of this approximation is that a good approximation to the eigenvalue is obtained. On the other hand, one cannot assume a priori that the approximate wave functions obtained in this manner are good approximations in all their details. Considerable caution must be exercised in the calculation of quantities dependent upon the detailed character of the wave function, such as magnetic moments.

3. – The Problem of the Center of Mass (Collective Translation).

A simple example of the common use of superfluous co-ordinates is the use of the full number of particle co-ordinates in the nuclear shell model, along with the assumption that the motion of the center of mass of the system has already been separated, so that the number of physical degrees of freedom of the system is reduced by three. This problem can be treated exactly by the formalism developed above, and is given here in full. But first a detailed discussion is given of certain general features of the problem, which are relevant to the treatment of more complicated collective motions, such as rotation and vibration. The points raised in this discussion define the essential similarities and differences between the more complicated cases and that of collective translation which is the simplest. A parallel discussion of these same features is given in section 4 for the rotation problem. In addition, an alternative treatment is presented which does not require the introduction of superfluous co-ordinates and deals with a set of independent co-ordinates. This is possible for the center-of-mass case because of the inherent simplicity of the problem. The significance of many of the peculiar features of the superfluous co-ordinate formalism is clarified by comparison with this alternative formulation.

3·1. *General features of the problem of the center of mass.* – Consider a system of particles subject only to forces acting between them. The Hamiltonian can be written in form

$$H = (1/2M) \sum_i (p_i)^2 + V, \tag{3.1}$$

and the following general remarks can be made:

1) *Invariance properties.* The form of the Hamiltonian (3.1) is invariant with respect to arbitrary translations of the co-ordinate system.

2) *Conservation law for collective motion variable.* The operator generating infinitesimal translations of the co-ordinate system in the direction α is $\sum_i p_{i\alpha}$. As a result of the invariance property (1), this operator commutes with the

Hamiltonian, and is a constant of the motion. $\sum_i p_{i\alpha}$ is the α-component of the total linear momentum of the system, and represents a mode of collective motion.

3) *Specification of collective motion.* The separation of the motion into two types, collective and intrinsic is of importance for this type of problem. One way to describe this division is to define a co-ordinate system which moves with the particles; in this case a co-ordinate system with its origin at the center of mass of the system. The motion of the particles with respect to the moving co-ordinate system is called the intrinsic motion; the motion *of the co-ordinate system* with respect to a co-ordinate system fixed in space is called the collective motion. The parameters describing the position of the moving axes are the collective co-ordinates (in this case, $\sum_i \boldsymbol{x}_i/A$, where A is the number of particles). The infinitesimal transformation operators corresponding to the mode of motion of the moving axes are momentum operators for the collective motion.

4) *Commutability of collective momentum operators.* All the infinitesimal translation operators $\sum_i p_{i\alpha}$ for different directions α commute with one another. Therefore a complete set of eigenfunctions of the operator H can be found which are also eigenfunctions of all the collective momentum operators $\sum p_{i\alpha}$.

5) *Separability of collective motion.* A transformation can be made from the individual particle co-ordinates $x_{i\alpha}$ (the latin index refers to particle number, the greek index to direction in space) to collective co-ordinates and a set of $3A-3$ independent intrinsic co-ordinates q_k, representing co-ordinates of particles with respect to the moving axes. These independent co-ordinates are difficult to work with in an actual problem, but a number of significant general conclusions can be based simply on the fact that such co-ordinates can be defined, without reference to their specific properties. One example of such a set of co-ordinates is the set of co-ordinates of $A-1$ of the particles with respect to the center of mass.

After the transformation is made, the Hamiltonian separates into two parts, a collective part $(\sum_i \boldsymbol{p}_i)^2/2AM$ and an intrinsic part depending only upon the q_k. The eigenfunctions of H can then be expressed as the product:

$$\psi_{Pn} = \varphi_n(q_k) \exp\left[i\boldsymbol{P}\cdot\sum_i \boldsymbol{x}_i/A\hbar\right], \tag{3.2}$$

where the vector number $\boldsymbol{P}$ is the eigenvalue of the vector operator $\sum_i \boldsymbol{p}_i$, the total linear momentum of the system, and $\varphi_n(q_k)$ is an eigenfunction of the *total* Hamiltonian H, depending only on the intrinsic co-ordinates q_k, described by the internal quantum number n, and having the energy value E_n. $\varphi_n(q_k)$ is

also an eigenfunction of the operator $\sum_i \boldsymbol{p}_i$ with eigenvalue zero. The energy value of the function $\psi_{\boldsymbol{P}n}$ is therefore

$$E_{\boldsymbol{P}n} = (\boldsymbol{P})^2/2AM + E_n \tag{3.3}$$

6) *Symmetry of Eigenfunctions.* The functions (3.2) are symmetric with regard to translations of the co-ordinates system, changing only by a phase factor. This symmetry follows from the uncertainty principle; since the total linear momentum of the system is known exactly, all positions in space are equally probable, and the system therefore is not localized within any finite region in space.

7) *Destruction of symmetry by self-consistent field approximation.* It is often convenient, when attempting to solve the Schrödinger equation for the system, to replace the potential acting between particles by a central potential fixed in space which is the sum of terms acting separately upon individual particles (e.g. the nuclear shell model). This destroys the symmetry of the Hamiltonian with respect to translations, and gives eigenfunctions which are localized in space and therefore cannot be exact eigenfunctions of $\sum_i \boldsymbol{p}_i$ nor of H. These functions can be interpreted as describing only the intrinsic motion of the system; collective motion must somehow be introduced in order to obtain an exact eigenfunction with the proper symmetry properties. Another way of describing these « shell-model functions » is as a linear combination of the real eigenfunctions (3.2), which if they are a good approximation involve only one intrinsic function φ_n and a « wave packet » of functions having different values of P. One can then write the « shell model function » ψ_{sn} corresponding to the intrinsic state φ_n as:

$$\psi_{sn} = \int \psi_{\boldsymbol{P}n} f(\boldsymbol{P})\, \mathrm{d}\boldsymbol{P}\,. \tag{3.4}$$

From equation (3.3) we see that the expectation value of the energy obtained with ψ_{sn} instead of the lowest energy state ψ_{0n} will be in error by an amount $P^2/2AM$, where the value of P is an average over the distribution $f(\boldsymbol{P})$. This shows, as one expects, that the shell model function gets better as the number of particles A increases.

8) *Non-existence of symmetrical bound states without collective motion.* – It is trivially evident that any state in which the particles are concentrated in a finite volume of space is not symmetrical with respect to translations, and therefore that any bound state must have collective motion to possess the required symmetry.

9) *Non-existence of bound states with relative linear momentum.* In a bound state, where the distance between any pair of particles cannot increase without

limit, the mean value of their relative linear momentum must vanish. Thus the total linear momentum of the system must all come from the collective motion, i.e. the linear momentum of the center of mass. It follows therefore that for any eigenstate of the Hamiltonian (3.1) corresponding to some value P of the total linear momentum, there exists an eigenstate corresponding to the same intrinsic motion (the same φ_n) but with total linear momentum zero.

10) *The «translational spectrum»*. For each intrinsic eigenstate φ_n, there is a spectrum of eigenstates of the Hamiltonian H, corresponding to different eigenvalues of the total linear momentum P, with energies given by (3.3). The spectrum is continuous.

A number of these features seem rather trivial, but their significance becomes apparent when more general types of collective motion are considered. This is evident when translation is replaced by rotation and linear momentum by angular momentum in the above statements as is done in section **4**. Some of these features continue to be true, others are definitely false, but none of them remain trivial.

3·2. *An alternative formulation.* – Let us consider, for the moment, a new problem with a Hamiltonian H' which is different from (3.1) and has the form:

$$H' = \sum_i (\boldsymbol{p}_i)^2/2M + V - (\sum_i \boldsymbol{p}_i)^2/2AM \,. \tag{3.5}$$

Since H' differs from H only by a function of $\sum_i \boldsymbol{p}_i$, any eigenfunction (3.2) of both H and of $\sum_i \boldsymbol{p}_i$ is also an eigenfunction of H'. The eigenvalue of H' however differs from that of H by the quantity $(\sum_i \boldsymbol{p}_i)^2/2AM = \boldsymbol{P}^2/2AM$. Thus the energy E' in our new problem is

$$E'_{\boldsymbol{P}n} = E_n \,, \tag{3.6}$$

instead of the value given by equation (3.3). $E'_{\boldsymbol{P}n}$ is therefore independent of $\boldsymbol{P}$; all eigenfunctions corresponding to the same intrinsic function φ_n are degenerate; and any linear combination of them is also an eigenfunction of H' corresponding to the energy E_n. The «shell model» function ψ_{sn} is therefore an eigenfunction of the operator H'. Furthermore the energy value E'_{sn} obtained by use of H' is just the value desired in the solution of problem (3.1), namely that of the lowest energy state corresponding to the intrinsic state φ_n. The alternative formulation thus defines a Hamiltonian operator which can be used with state functions localized in space and yet give correct

energy values. Note again that the additional term varies as $1/A$, and is therefore small when A is large. It can be interpreted as the kinetic energy required by uncertainty principle for a system of mass AM which is confined to a finite region of space.

By a procedure similar to the construction of H', other operators B' can be constructed having the property of acting on the shell model functions in the same way as an operator B acts on the lowest energy eigenfunction ψ_{0n} corresponding to the intrinsic state φ_n. The operator B' must satisfy the following two conditions:

1) B' operates only on the q_k, i.e. it commutes with the operators $\sum \boldsymbol{x}_i$ and $\sum \boldsymbol{p}_i$.

2) The operator $B' - B$ must give zero when operating on all functions ψ_{0n} corresponding to total linear momentum zero. It is then evident that

$$(3.7)\qquad \frac{\int \psi^*_{sm} B' \psi_{sn}\, \mathrm{d}\boldsymbol{x}_i}{\left[\int \psi^*_{sm}\psi_{sm}\,\mathrm{d}\boldsymbol{x}_i \cdot \int \psi^*_{sn}\psi_{sn}\,\mathrm{d}\boldsymbol{x}_i\right]^{\frac{1}{2}}} \frac{\int \varphi^*_m(q_k) B \varphi_n(q_k)\mathrm{d}q_k}{\left[\int \varphi^*_m(q_k)\varphi_m(q_k)\,\mathrm{d}q_k \cdot \int \varphi^*_n(q_k)\varphi_n(q_k)\,\mathrm{d}q_k\right]^{\frac{1}{2}}} = = \frac{\int \psi^*_{0m} B \psi_{0n}\,\mathrm{d}\boldsymbol{x}_i}{\left[\int \psi^*_{0m}\psi_{0m}\,\mathrm{d}\boldsymbol{x}_i \cdot \int \psi^*_{0n}\psi_{0n}\,\mathrm{d}\boldsymbol{x}_i\right]^{\frac{1}{2}}}.$$

As an example of such an operator, consider the angular momentum of the i-th particle with respect to the center of mass of the system:

$$(3.8)\qquad \boldsymbol{L}_i = (\boldsymbol{x}_i - \sum_j \boldsymbol{x}_j/A) \times \boldsymbol{p}_i\,.$$

A corresponding operator $\boldsymbol{L}'_i$ which can be used on shell model functions is

$$(3.9)\qquad \boldsymbol{L}'_i = (\boldsymbol{x}_i - \sum_j \boldsymbol{x}_j/A) \times (\boldsymbol{p}_i - \textstyle\sum_k \boldsymbol{p}_k/A)\,.$$

The operator $\boldsymbol{L}'_i$ commutes with $\sum \boldsymbol{x}_i$ and is equal to $\boldsymbol{L}_i$ when operating on functions of total linear momentum zero. $\boldsymbol{L}'_i$ can then be used for calculations with shell model functions ψ_{sn}. The total orbital magnetic moment of a system of Z protons and $A-Z$ neutrons is obtained by taking the sum of $g_L \boldsymbol{L}'_i$ over all the protons in the system, where g_L is the appropriate « g-factor » for a proton. For a system in which all the protons are in closed shells, and there is only a single neutron outside closed shells, the total orbital magnetic moment would be zero if one neglected the motion of the odd neutron and the rest of the nucleus about the common center of mass. This effect is automatically included if one uses the operator $\boldsymbol{L}'_i$ on the shell model wave functions.

Since the sum is over all the protons and they are in closed shells, most of the terms in (3.9) vanish after the summation, as they are of the type $x_{i\alpha}p_{j\beta}$. The only non-vanishing terms are those in which both i and j refer to the odd neutron, and these give the angular momentum of the neutron which is equal to the total angular momentum of the nucleus $\boldsymbol{I}$. Since these terms appear with the coefficient $1/A^2$ and there is one such term for each proton, the result obtained for the orbital magnetic moment is

$$\boldsymbol{\mu}_L = g_L \boldsymbol{I}(Z/A^2) \,. \tag{3.10}$$

3·3 *Superfluous Co-ordinate Formulation.* – Let us now consider the problem of the center of mass, making use of the superfluous co-ordinates. This can be done by introducing an extra vector co-ordinate $\boldsymbol{x}_0$, and defining the following transformation to co-ordinates $\boldsymbol{\xi}_k$ $(0 \leqslant k \leqslant A)$:

$$\boldsymbol{x}_i = \boldsymbol{\xi}_i - \boldsymbol{\xi}_0 \,, \tag{3.11}$$

$$\boldsymbol{x}_0 = -\sum_1^A \boldsymbol{\xi}_i \,. \tag{3.11'}$$

The inverse transformation is given by:

$$\boldsymbol{\xi}_i = \boldsymbol{x}_i - (1/A)\sum_0^A \boldsymbol{x}_k \,, \tag{3.12}$$

$$\boldsymbol{\xi}_0 = -(1/A)\sum_0^A \boldsymbol{x}_k \,. \tag{3.12'}$$

When $\boldsymbol{x}_0$ is zero, $-\boldsymbol{\xi}_0$ is clearly equal to the co-ordinate of the center of mass of the system. The $\boldsymbol{\xi}_i$ are particle co-ordinates with respect to a co-ordinate system whose origin is located at a point having the co-ordinates $-\boldsymbol{\xi}_0$ in the x system. The negative sign is arbitrarily chosen for convenience in writing summations.

From equation (3.12) one can immediately write the operator:

$$\sum_{i=1}^A p_{i\alpha}^2 = -\hbar^2 \sum_{i=1}^A \partial^2/\partial x_{i\alpha}^2 = \sum_{i=1}^A \sum_{k=0}^A \sum_{m=0}^A \frac{\partial \xi_{k\alpha}}{\partial x_{i\alpha}} \pi_{k\alpha} \frac{\partial \xi_{m\alpha}}{\partial x_{i\alpha}} \pi_{m\alpha} = \tag{3.13}$$

$$= \sum_{i=1}^A \sum_{k=0}^A \sum_{m=0}^A (\delta_{ki} - 1/A)(\delta_{mi} - 1/A)\pi_{k\alpha}\pi_{m\alpha} = \sum_{i=1}^A \pi_{i\alpha}^2 + (1/A)\pi_{0\alpha}^2 - (1/A)\sum_{k=1}^A \sum_{m=1}^A \pi_{k\alpha}\pi_{m\alpha} \,,$$

where

$$\pi_{i\alpha} = (\hbar/i)\partial/\partial \xi_{i\alpha} \,.$$

The Hamiltonian (3.1) in the new co-ordinates is thus

$$H'' = (\boldsymbol{\pi}_0)^2/2AM + \sum_{i=1}^{A} (\boldsymbol{\pi}_i)^2/2M + V(\boldsymbol{\xi}_i) - (\sum \boldsymbol{\pi}_i)^2/2AM \,. \tag{3.14}$$

The potential V depends only upon distances between particles, and from equation (3.11) $\boldsymbol{x}_i - \boldsymbol{x}_j = \boldsymbol{\xi}_i - \boldsymbol{\xi}_j$. It follows therefore that V has the same functional form in (3.14) as in (3.1).

The co-ordinates $\boldsymbol{\xi}_0$ are clearly separable in (3.14), so that the eigenfunctions can be written in the form

$$\psi_{\boldsymbol{P}n} = \varphi_n(\boldsymbol{\xi}_i) \exp\left[i\boldsymbol{P}\cdot\boldsymbol{\xi}_0/\hbar\right], \tag{3.15}$$

corresponding to the eigenvalue P of the operator $\boldsymbol{\pi}_0$, and characterized by other quantum numbers n. $\varphi_n(\boldsymbol{\xi}_i)$ is an eigenfunction of (3.14) and of $\boldsymbol{\pi}_0$, corresponding to eigenvalues E_n and 0 respectively. Thus φ_n is an eigenfunction of the operator

$$H''' = \sum_i (\boldsymbol{\pi}_i)^2/2M + V - (\sum_i \boldsymbol{\pi}_i)^2/2AM \,, \tag{3.16}$$

H''' has the same number of co-ordinates as the original Hamiltonian, (3.1) even though three degrees of freedom have already been removed. The operators $\sum \pi_{i\alpha}$ commute with H''' and seem to define new constants of the motion and quantum numbers associated with some sort of collective translation, in addition to the motion of the center of mass which has already been removed. These peculiarities are due to the introduction of the superfluous co-ordinates, and can be understood easily after noting that the Hamiltonian (3.16) is of exactly the same form as the Hamiltonian H' given by (3.5), with latin letters replaced by greek. All the conclusions of section 3.2 can therefore be applied immediately, and the eigenfunctions of (3.16) can therefore be written in the form

$$\psi_{\boldsymbol{\beta}n} = \varphi_n(\varkappa_k) \exp\left(i\boldsymbol{\beta}\cdot\sum \boldsymbol{\xi}_i/A\hbar\right), \tag{3.17}$$

where $\boldsymbol{\beta}$ is a new quantum number and $\varkappa_i$ are a set of independent combinations of the ξ_i analogous to the q_i in the preceeding case. Although for each function φ_n there are an infinite number of eigenfunctions $\psi_{\boldsymbol{\beta}n}$ corresponding to different values of $\boldsymbol{\beta}$, the eigenvalue of H''' is independent of $\boldsymbol{\beta}$. Furthermore, the operation of all *physical* operators on $\psi_{\boldsymbol{\beta}n}$ is also independent of $\boldsymbol{\beta}$, since $\exp\left[i\boldsymbol{\beta}\sum\boldsymbol{\xi}_i\right]$ is just $\exp\left[-i\boldsymbol{\beta}\boldsymbol{x}_0\right]$, and a physical operator by definition does not contain the extra co-ordinate x_0 or derivatives with respect to it. If we now construct a physical state function from (3.17) by multiplying by $[\delta(\boldsymbol{x}_0)]^{\frac{1}{2}}$, we see that all functions $\psi_{\boldsymbol{\beta}n}$ having the same function φ_n correspond to the same physical state function, regardless of the value of $\boldsymbol{\beta}$.

The result of the introduction of the extra co-ordinate in this case is thus to introduce an extraneous quantum number corresponding to an apparent collective motion having no physical significance, and of which all eigenvalues and expectation values of physical operators are independent. Because of this degeneracy all physical quantities can be calculated using any one particular eigenfunction of this extraneous operator, or any convenient linear combination of its eigenfunctions. One such combination is the « shell model function », obtained by replacing the inter-particle forces by an equivalent self-consistent field in the space of the ξ co-ordinates.

4. – The Problem of Collective Rotational Motion.

Superfluous co-ordinates can be defined and used to treat the problem of collective rotation in a manner similar to that used in the preceding section to treat collective translation. The rotation problem is, however, considerably more complicated, and the reasons for this are indicated by a consideration of those same general features of the problem discussed in section 3 for the translational case.

4·1. *General features of the collective rotation problem.* – Consider again a system of particles subject only to forces acting between particles. The Hamiltonian has the form (3.1) as in the translational case, and the same general features discussed there can be examined here. They are listed below in the same order and with the same headings so that they can be compared directly with section 3. The points of similarity and difference outline the general characteristics of the rotation problem.

1) *Invariance properties.* The form of the Hamiltonian (3.1) is invariant with respect to arbitrary rotations of the co-ordinate system.

2) *Conservation law for collective motion variable.* The operator generating infinitesimal rotations of the co-ordinate system about a given axis is the component of the total angular momentum vector L of the system in the direction of the given axis. As a result of the invariance property (1), this operator commutes with the Hamiltonian and is a constant of the motion. L_α is the α-component of the total angular momentum of the system and *may* (see 9 below) represent a mode of collective motion.

3) *Specification of Collective Motion.* The motion can be divided into two types, collective and intrinsic by defining a co-ordinate system which moves with the particles. In this case the motion is one of rotation and is specified by collective co-ordinates (e.g. Euler angles) giving the location of the moving axes with respect to axes fixed in space. The intrinsic motion is described by

individual particle co-ordinates with respect to the moving axes. Unlike the case of the center of mass, there is no a priori method of defining the new axes with respect to the particles in a natural way. A variety of definitions may be used (e.g. the principal axes of inertia, a set of axes determined by the octupole moment, etc.), and the value of each depends upon the particular problem involved.

4) *Commutability of Collective Motion Operators.* Infinitesimal rotation operators about different axes do not commute. In choosing a complete set of eigenfunctions for the system it is necessary to pick a definite axis α, and to use simultaneous eigenfunctions of L_α and L^2.

5) *Separability of collective motion.* The separation of collective motion from the intrinsic motion is not automatic, as in the translation case. WIGNER [5] has shown that a wave function for a system of particles having total angular momentum L and projection M upon a fixed axis can always be written in the following form:

$$\psi_M^L(x_i) = \sum_K D_{MK}^{L*}(\alpha_j)\varphi_K(q_k)\,, \tag{4.1}$$

where α_j are the Euler angles defining the position of a system of axes moving with the particles, and q_k are intrinsic co-ordinates of the particles with respect to the moving axes. The functions D_{MK}^{L*} are the proper functions for the symmetric top. The form of the functions φ_K depends upon the Hamiltonian of the system and upon the way in which the mowing co-ordinate system is defined with respect to the particles. A true separation of the intrinsic and collective motion is achieved if the axes can be defined in such a way that all of the functions φ_K in equation (4.1) are proportional to one another and the wave function can be written as the product of two factors depending respectively only upon α_j and q_k. The energy of such a separated eigenfunction can be expressed as the sum of a collective and an intrinsic part, but these two are no longer independent. The collective motion is that of an asymmetric rotator whose moments of inertia depend upon the form of the intrinsic wave function.

If all of the φ_K in equation (4.1) can be made to vanish except one, the resulting collective wave function has the properties of the symmetric rotator. This is the case of chief interest. The sum in equation (4.1) then reduces to a single term [6].

6) *Symmetry of eigenfunctions.* In contrast to the translational case,

[5] E. WIGNER: *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Braunschweig, 1931), p. 228.

[6] Two terms are actually necessary for symmetry reasons, as has been pointed out by BOHR and MOTTELSON (see ref. [1]), but this distinction is not important here.

the functions (4.1) do not in general represent a spherically symmetric density distribution; i.e. they change by more than just a phase factor when the coordinate system is rotated. The only completely symmetric function is that of the state of total angular momentum zero. The others have the transformation properties of an irreducible tensor of degree L.

The difference from the translational case is due to the non-commutativity of the infinitesimal rotation operators. A wave function which is an eigenfunction of L_z does possess the property of changing only by a phase factor as a result of rotations about the z-axis. This, as in the translational case can be associated with the uncertainty principle; since the angular momentum about the z-axis is known exactly, the canonically conjugate angle variable is not known at all and all values are equally probable. However, the fact that the state is an eigenfunction of L_z, means that unless $L = 0$, the state is not an eigenfunction of L_y or L_x, and therefore is not symmetric about these axes.

7) *Destruction of symmetry by self-consistent field approximation.* If the potential between particles is replaced by a single particle potential fixed in space, the symmetry of the Hamiltonian with respect to rotations may or may not be destroyed, depending upon the form of the potential. In the event that the potential is not spherically symmetric (deformed), the symmetry is destroyed, and eigenfunctions are obtained which do not possess the required transformation properties and are therefore not eigenfunctions of L^2 nor of the original Hamiltonian H. These functions can be interpreted as describing only the intrinsic motion of the system, and collective motion must be added to obtain the required transformation properties. However, the possibility of spherically symmetric potentials also exists. These potentials are invariant with respect to rotation and therefore solutions can be found having the required transformation properties. This possibility does not exist in the translation case because a potential invariant with respect to translations can only be a constant throughout all space.

8) *Existence of symmetrical bound states.* In contrast to the translational case, rotational invariance is not incompatible with the concentration of the particles in a definite volume of space. It is therefore possible to find intrinsic wave functions for bound states which have the required transformation properties.

9) *Existence of bound states with relative angular momentum.* In contrast to the translational case, the average value of the angular momentum of one particle with respect to the others need not vanish in order to insure that the distance between particles remains bounded. There exists therefore the possibility of states having intrinsic angular momentum. The total angular mo-

mentum of such states is the sum of the intrinsic and collective angular momenta.

10) *The Rotational Spectrum.* For each eigenstate φ_K of the intrinsic motion one can define a series of functions (4.1), provided that φ_K itself is not spherically symmetric. This series of functions will be all eigenfunctions of the Hamiltonian in the event that: 1) the variables are separable; and 2) the adiabatic approximation is valid, i.e. the form of the intrinsic part of the wave function is approximately independent of the rotational quantum numbers. A discrete set of eigenvalues is obtained, due to the quantization of angular momentum, with a spacing depending upon certain properties of the intrinsic part of the wave function, i.e. the effective moment of inertia.

Because of the differences between the rotational and translational cases pointed out by 6-9, it is convenient to divide the rotational problem into three cases.

Case I. Deformed nuclei having total angular momentum zero in the ground state. This is the case closely analogous to the translational case. The symmetry of the lowest rotational state results from collective motion and the total angular momentum in the rotational states is all due to collective motion.

Case II. Spherical Nuclei. The symmetry is intrinsic and there is no collective rotation or rotational spectrum.

Case III. Nuclei of non-zero total angular momentum in the ground state. The intrinsic angular momentum is finite and the lowest rotational state is not spherically symmetric. The total angular momentum is always the resultant of intrinsic and collective angular momenta.

4·2. *Superfluous Co-ordinate Formulation with Physical Hamiltonian.* – Let us now consider the problem of rotation making use of superfluous co-ordinates. As in the case of the center of mass, an extra vector co-ordinate $x_{0\sigma}$ can be introduced, and the transformation to new co-ordinates ξ_{kx} can be defined as follows (2):

$$\begin{cases} x_{i\mu} = [a_{\alpha\mu}(\xi_{0\beta})]\xi_{i\alpha}\,, \\ x_{0\sigma} = x_{0\sigma}(\xi_{i\alpha})\,. \end{cases} \tag{4.2}$$

The matrix $a_{\alpha\mu}$ is an orthogonal matrix having a determinant unity whose explicit form as a function of the parameters ξ_0 is well known (7). A specific definition of the Euler angles is not necessary for the purpose of this work,

(7) E. T. WHITTAKER: *Analytical Dynamics* (Cambridge, 1937), p. 10

the general form (4.2) being sufficient. The functional relation $x_{0\sigma}(\xi_{i\alpha})$ can be chosen to meet the needs of the specific problem. It determines the way in which the moving co-ordinate system is defined with respect to the particles.

Because of the dependence of the $a_{\alpha\mu}$ on the collective co-ordinates, the inverse transformation of (4.2) cannot be written explicitely even for definite functions $x_{0\sigma}(\xi_{i\alpha})$. The direct evaluation of the momenta $p_{i\mu}$ in terms of the ξ's is somewhat involved. A simpler way to obtain the momenta depends upon the use of geometrical considerations and of infinitesimal rotation operators.

The $a_{\alpha\mu}$ are three mutually perpendicular unit vectors in the space of the $x_{i\mu}$, μ being the running index, and α being a label for the vector. Since these vectors are functions of the co-ordinates x_i (the superfluous co-ordinates may be considered here as parameters unaffected by geometrical rotations of the real co-ordinate system), the $a_{\alpha\mu}$ satisfy the usual commutation rules for vectors with the components of the total angular momentum vector

$$\hbar L_\mu = \sum_{\nu\varkappa} \varepsilon_{\mu\nu\varkappa} x_{i\nu} p_{i\varkappa} ,$$

of the system of particles:

$$[L_\mu, a_{\alpha\nu}] = i \sum_{\varkappa} \varepsilon_{\mu\nu\varkappa} a_{\alpha\varkappa} , \tag{4.3}$$

where $\varepsilon_{\mu\nu\varkappa}$ is the antisymmetric tensor of rank three whose elements are plus one if $\mu\nu\varkappa$ is an even permutation of 1, 2, 3, and minus one if it is an odd permutation. The projections of the angular momentum vector L_μ on the three vectors $a_{\alpha\mu}$ are now defined:

$$\Lambda_\alpha = a_{\alpha\mu} L_\mu . \tag{4.4}$$

The definition (4.4) being a scalar product of the two vectors $a_{\alpha\mu}$ and L_μ, the operators Λ_α are *scalars* in the space of the x_i, and therefore commute with $\boldsymbol{L}$.

By use of the relations (4.3), (4.4) and the properties of the matrices $a_{\alpha\mu}$

$$\sum_{\mu} a_{\beta\mu} a_{\alpha\mu} = \delta_{\beta\alpha}; \qquad \sum_{\alpha} a_{\alpha\mu} a_{\alpha\nu} = \delta_{\mu\nu}; \qquad \det \|a\| = 1 , \tag{4.5}$$

the following relations are directly obtained:

$$\begin{cases} [\Lambda_\alpha, L_\mu] = 0 & [\Lambda_\alpha, \Lambda_\beta] = -i \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} \Lambda_\gamma , \\ [\Lambda_\alpha, a_{\beta\mu}] = -i \sum_{\gamma} \varepsilon_{\alpha\beta\gamma} a_{\gamma\mu} & \sum_{\alpha} (\Lambda_\alpha)^2 = \sum_{\mu} (L_\mu)^2 . \end{cases} \tag{4.6}$$

Thus the operators Λ_α and $a_{\alpha\mu}$ have properties of vectors in the three indices α, and the operators Λ_α act as infinitesimal rotations in a negative direction in this space. Furthermore, since the L_μ and the Λ_α are infinitesimal rotation operators in the x_i space which rotate the co-ordinate axes while leaving the particles and the vectors $\boldsymbol{a}_i$ fixed in space, they commute with all the intrinsic co-ordinates $\xi_{i\alpha}$ $(i \neq 0)$ and act only on the co-ordinates $\xi_{0\sigma}$ in the ξ representation. The operators Λ_α are thus equivalent to the operators Q_i defined by CASIMIR [8] in the theory of the « spinning top ». Casimir's results can therefore be taken over directly for that part of the problem concerned with the operation of the operators Λ_α on functions of the Euler angles $\xi_{0\alpha}$.

Let us now define also an operator of infinitesimal rotations in the ξ space:

$$\hbar\lambda_\alpha = \sum_{i\beta\gamma} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}\pi_{i\gamma} = (\hbar/i)\sum_{i\beta\gamma} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}\partial/\partial\xi_{i\gamma}\,, \tag{4.7}$$

where $\pi_{i\gamma} = (\hbar/i)\partial/\partial\xi_{i\gamma}$, and the summation on i excludes $i = 0$. All further summations on particle indices exclude the index 0.

The momenta p_i can now be obtained in terms of the ξ's and the operators Λ_α and λ_α. From equation (4.2),

$$\pi_{i\alpha} = \sum_{j\mu} (\partial x_{j\mu}/\partial\xi_{j\alpha})p_{j\mu} + \sum_\sigma (\partial x_{0\sigma}/\partial\xi_{i\alpha})p_{0\sigma} = \sum_\mu a_{\alpha\mu}p_{i\mu} + \sum_\sigma (\partial x_{0\sigma}/\partial\xi_{i\alpha})p_{0\sigma}. \tag{4.8}$$

Substituting (4.8) into the definition (4.7) of λ_α, and using the relations (4.5) and the definition (4.4) of Λ_α we obtain

$$\hbar\lambda_\alpha = \sum_{i\beta\gamma\mu} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}a_{\gamma\mu}p_{i\mu} + \sum_{\sigma i\beta\gamma} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}(\partial x_{0\sigma}/\partial\xi_{i\mu})p_{0\sigma} = \hbar\Lambda_\alpha + i\sum_\sigma [\lambda_\alpha, x_{0\sigma}]p_{0\sigma}\,. \tag{4.9}$$

The commutator $[\lambda_\alpha, x_{0\sigma}]$ is a square matrix in the indices α and σ. We can therefore write

$$X_{\alpha\sigma} = i[\lambda_\alpha, x_{0\sigma}]\,. \tag{4.10}$$

If the determinant of $X_{\alpha\sigma}$ does not vanish, the inverse matrix $(X)^{-1}_{\sigma\alpha}$ is defined, and equation (4.9) can be written:

$$p_{0\sigma} = \hbar\sum_\alpha (X)^{-1}_{\sigma\alpha}(\lambda_\alpha - \Lambda_\alpha)\,. \tag{4.11}$$

Substituting (4.11) into (4.8),

$$\sum_\mu a_{\alpha\mu}p_{i\mu} = \pi_{i\alpha} + \hbar\sum_{\sigma\beta} (\partial x_{0\sigma}/\partial\xi_{i\alpha})(X)^{-1}_{\sigma\beta}(\Lambda_\beta - \lambda_\beta)\,, \tag{4.12}$$ [9]

[8] H. B. G. CASIMIR: *Rotation of a Rigid Body in Quantum Mechanics, Diss.* (Groningen, 1931).

or

$$p_{i\mu} = \sum_{\alpha} a_{\alpha\mu}\pi_{i\alpha} + \hbar \sum_{\alpha\beta\sigma} a_{\alpha\mu}(\partial x_{0\sigma}/\partial\xi_{i\alpha})(X)^{-1}_{\sigma\beta}(\Lambda_\beta - \lambda_\beta)\,. \tag{4.13}$$

Noting that $a_{\alpha\mu}$ depends only on the Euler angles ξ_0 and commutes with the π_β and that $p_{i\mu}$ commutes with $p_{0\sigma}$, the kinetic energy operator $T = (1/2M)\sum_{i\mu} p^2_{i\mu}$ (M being the mass of each particle) can be written

$$2MT = \sum_{\alpha\mu i} \pi_{i\alpha}a_{\alpha\mu}p_{i\mu} + \sum_{i\alpha\mu\sigma} a_{\alpha\mu}(\partial x_{0\sigma}/\partial\xi_{i\alpha})p_{i\mu}p_{0\sigma} = \tag{4.14}$$

$$= \sum_{\alpha i}\pi^2_{i\alpha} + \hbar^2 \sum_{\substack{i\alpha\sigma\\ \beta\nu\gamma}} (\partial x_{0\sigma}/\partial\xi_{i\alpha})(\partial x_{0\nu}/\partial\xi_{i\alpha})(X)^{-1}_{\sigma\beta}(\Lambda_\beta - \lambda_\beta)(X)^{-1}_{\nu\gamma}(\Lambda_\gamma - \lambda_\gamma) +$$

$$+ 2\hbar \sum_{i\alpha\beta\sigma} (\partial x_{0\sigma}/\partial\xi_{i\alpha})\pi_{i\alpha}(X)^{-1}_{\sigma\beta}(\Lambda_\beta - \lambda_\beta) + (\hbar^2/i)\sum_{i\alpha\beta\sigma} (\partial^2 x_{0\sigma}/\partial\xi^2_{i\alpha})(X)^{-1}_{\sigma\beta}(\Lambda_\beta - \lambda_\beta)\,.$$

The Hamiltonian is obtained by adding the potential energy to the kinetic energy (4.14). Since the potential is assumed to be due only to inter-particle forces, its form is unchanged by the transformation which is only a rotation of co-ordinate axes. One would hope to solve the problem by using shell model techniques; namely to replace the interparticle potential by a self-consistent field approximation consisting of the sum of single-particle terms fixed in space but not necessarily spherically symmetric. If the intrinsic and collective motions are separable, the intrinsic part of the wave function may be given by some sort of shell model, possibly deformed, which rotates in a manner described by the collective part of the wave function. The rotational energy is then determined by the rotational quantum numbers and by the effective moment of inertia, which is in turn determined by the intrinsic part of the wave function.

A popular form for the potential is the anisotropic harmonic oscillator, as has been considered by INGLIS ([10]), who used a classical description of the rotation with angular velocity Ω to avoid the necessity of introducing extra co-ordinates. Using the Hamiltonian (4.14), one would expect to calculate the intrinsic part of the wave function as INGLIS has done, but to describe the rotation in quantum-mechanical terms.

([9]) Equation (4.12) can be interpreted as giving the projections of the particle momentum vector $p_{i\mu}$ on the axes of the intrinsic motion, and as being the sum of two terms representing intrinsic and collective momentum ([2]), the term proportional to Λ_β being the collective momentum. The coefficient of Λ_β is a function of the ξ_i and under certain conditions the value of this coefficient $(\partial x_{0\sigma}/\partial\xi_{i\alpha})(X)^{-1}_{\sigma\beta}$ averaged over all the co-ordinates ξ_j ($j\neq i$) can be considered as defining a velocity field for i-th particle.

([10]) D. R. INGLIS: *Phys. Rev.*, **96**, 1059 (1954).

In such a description the operators λ_α perform infinitesimal rotations on the intrinsic part of the wave function, and might be considered as « intrinsic » angular momenta. Since the potential is deformed, λ^2 is not a good quantum number but if there is a symmetry axis β, then λ_β is a good quantum number. The operators Λ_α can be identified with the total angular momentum of the system, since they are linear combinations of the total angular momentum components L_μ, such that $\Lambda^2 = L^2$. The rotational angular momentum is thus $(\Lambda_\alpha - \lambda_\alpha)$; and as the operators Λ appear always in this combination in the kinetic energy, one can interpret some of these terms as being rotational energy and others as coupling energy between the rotational and intrinsic motion.

It should however be noted that although Λ_α is a semi-physical operator and $\Lambda^2 = L^2$ is a physical operator, λ_α is neither physical nor semi-physical. The exact nature of the terms involving λ_α in the kinetic energy is not evident, as in the translational case; however one might by analogy expect to find a degeneracy of states having different values of λ^2. In this case a linear combination of such states can also be an eigenfunction of the Hamiltonian. This combination is no longer an eigenfunction of λ^2 and may therefore represent a wave function of a deformed potential.

If the intrinsic and collective motions are separable, and if the adiabatic approximation is valid, it is convenient to split the kinetic energy into three parts, as has been done by A. Bohr [1], a rotational energy T_{rot}, an intrinsic energy T_{int}, and a coupling energy T_{coupl} which should be small or zero. The rotational energy has the form of the rotational energy of a rigid body:

$$T_{\text{rot}} = \hbar^2 \sum_{\alpha\beta} A_{\alpha\beta}(\Lambda_\alpha - \bar{\lambda}_\beta)(\Lambda_\alpha - \bar{\lambda}_\beta)\,, \tag{4.15}$$

where $\bar{\lambda}_\alpha$ and $\bar{\lambda}_\beta$ are the averages of these operators over the intrinsic part of the wave function and represent the mean intrinsic angular momentum. The quantity $\Lambda_\alpha - \bar{\lambda}_\alpha$ is thus the mean rotational angular momentum. The reciprocal tensor of inertia $A_{\alpha\beta}$ is given by:

$$A_{\alpha\beta} = (1/2M) \sum_{i\gamma\sigma\nu} (\partial x_{0\sigma}/\partial \xi_{i\gamma})(\partial x_{0\nu}/\partial \xi_{i\gamma})(X)^{-1}_{\sigma\alpha}(X)^{-1}_{\nu\beta}\,. \tag{4.16}$$

There is no approximation involved in writing T_{rot} in this form, since T_{int} and T_{coupl} can be written so that the sum of the three terms is rigorously equal to (4.14). Thus:

$$2MT_{\text{int}} = \sum_{i\alpha} \pi^2_{i\alpha} + \hbar^2 \sum_{\substack{i\alpha\sigma \\ \nu\beta\gamma}} (\partial x_{0\sigma}/\partial \xi_{i\alpha})(\partial x_{0\nu}/\partial \xi_{i\alpha})(X)^{-1}_{\sigma\beta}[\lambda_\beta(X)^{-1}_{\nu\gamma}\lambda_\gamma - \bar{\lambda}_\beta(X)^{-1}_{\nu\gamma}\bar{\lambda}_\gamma] \tag{4.17}$$

$$- 2\hbar \sum_{\sigma i\alpha\beta} (\partial x_{0\sigma}/\partial \xi_{i\alpha})\pi_{i\alpha}(X)^{-1}_{\sigma\beta}(\lambda_\beta - \bar{\lambda}_\beta) - (\hbar^2/i) \sum_{i\alpha\beta\sigma} (\partial^2 x_{0\sigma}/\partial^2 \xi_{i\alpha})(X)^{-1}_{\sigma\beta}(\lambda_\beta - \bar{\lambda}_\beta)\,,$$

and

$$(4.18)\quad 2MT_{\text{coupl}} = -\hbar^2 \sum_{\substack{i\alpha\beta \\ \sigma\varkappa\gamma}} (\partial x_{0\sigma}/\partial\xi_{i\alpha})(\partial x_{0\varkappa}/\partial\xi_{i\alpha})(X)^{-1}_{\sigma\beta}[\Lambda_\beta(X)^{-1}_{\varkappa\gamma}(\lambda_\gamma - \bar{\lambda}_\gamma) +$$

$$+ (\lambda_\beta - \bar{\lambda}_\beta)(X)^{-1}_{\varkappa\gamma}\Lambda_\gamma] + 2\hbar \sum_{i\alpha\beta\sigma} (\partial x_{0\sigma}/\partial\xi_{i\alpha})\pi_{i\alpha}(X)^{-1}_{\sigma\beta}\Lambda_\beta + (\hbar^2/i)\sum_{i\alpha\beta\sigma} (\partial^2 x_{0\sigma}/\partial\xi^2_{i\alpha})(X)^{-1}_{\sigma\beta}\Lambda_\beta .$$

There is a certain arbitrariness in distributing certain of the terms between T_{rot}, T_{int}, and T_{coupl}. The choice made above is largely a matter of convenience, placing desired terms in T_{rot} and unpleasant ones like the linear terms in Λ_β in T_{coupl}. The rotational energy thus has the form given by CASIMIR [11].

At this point the adiabatic approximation can be introduced by replacing all quantities in T_{rot} and T_{coupl} which depend upon the intrinsic variables by their mean values. In particular the reciprocal moment of inertia $A_{\alpha\beta}$ given by equation (4.16) is replaced by its mean value $\bar{A}_{\alpha\beta}$.

The case of greatest interest is that giving rise to rotational spectra of a simple form. This is obtained when the component of the intrinsic angular momentum in a certain direction is a good quantum number. Call this component λ_3: Then $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$. If the transformation is chosen to make $\bar{A}_{\alpha\beta}$ diagonal, then $\bar{A}_{11} = \bar{A}_{22}$ because of the symmetry of the intrinsic wave function. In this case T_{rot} has the particularly simple form:

$$(4.15')\qquad T_{\text{rot}} = \hbar^2\bar{A}_{11}(\Lambda^2 - \Lambda_3^2) + \bar{A}_{33}(\Lambda_3 - \lambda_3)^2 .$$

The wave functions are simultaneous eigenfunctions of Λ_3, λ_3, and Λ^2. A rotational spectrum is a series of levels having the same values of Λ_3 and λ_3, but different values of $\Lambda^2 = L^2$. The energy levels spacings are thus proportional to $L(L+1)$.

The crucial point of this treatment is the validity of the separation of the intrinsic and collective motions which depends upon the relative contribution of T_{coupl}. The form of the transformation, determined by the relations expressing the superfluous co-ordinates $x_{0\sigma}$ in terms of the ξ's should be chosen to minimize the effect of T_{coupl}.

4·3. *The Bohr Transformation.* – A special case of collective rotational motion has been described by A. BOHR [2], and is obtained from the general formalism above by setting

$$(4.19)\qquad x_{0\sigma} = \tfrac{1}{2}\sum_{i\alpha\beta} |\varepsilon_{\sigma\alpha\beta}|\,\xi_{i\alpha}\xi_{i\beta} ,$$

where $|\varepsilon_{\sigma\alpha\beta}| = (1-\delta_{\sigma\alpha})(1-\delta_{\alpha\beta})(1-\delta_{\beta\sigma})$. The subsidiary condition $x_{0\sigma} = 0$ thus means that the moving co-ordinate axes are the principal axes of inertia

[11] H. B. G. CASIMIR: loc. cit. section V, 3.

of the system of particles. Then

$$\partial x_{0\sigma}/\partial \xi_{i\alpha} = \sum_{\beta} |\varepsilon_{\sigma\alpha\beta}| \xi_{i\beta} , \tag{4.20}$$

and

$$i[\lambda_\gamma, x_{0\sigma}] = \sum_{i\alpha\beta\delta} \varepsilon_{\gamma\delta\alpha}\xi_{i\delta} |\varepsilon_{\sigma\alpha\beta}| \xi_{i\beta} = \sum_{\alpha\beta i} \delta_{\gamma\sigma}\varepsilon_{\sigma\beta\alpha}\xi_{i\beta}^2 + \sum_{\alpha} \varepsilon_{\gamma\sigma\alpha}x_{0\sigma} . \tag{4.21}$$

Substituting equation (4.21) into equation (4.9), we obtain

$$p_{0\alpha} = \frac{1}{\sum_{\beta\gamma} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}^2} (\hbar\lambda_\alpha - \hbar\Lambda_\alpha - \sum_{\sigma\gamma} \varepsilon_{\alpha\sigma\gamma}x_{0\gamma}p_{0\sigma}) = \frac{\hbar}{\sum_{\beta\gamma} \varepsilon_{\alpha\beta\gamma}\xi_{i\beta}^2} (\lambda_\alpha - \Lambda_\alpha + \mathcal{L}_\alpha) , \tag{4.22}$$

where $\sum_{\beta\gamma} \varepsilon_{\alpha\beta\gamma}x_{0\beta}p_{0\gamma}/\hbar$ is denoted by $\mathcal{L}_\alpha$ because of its formal resemblance to an angular momentum operator. Equation (4.22) does not really give $p_{0\nu}$ in terms of the ξ's, since the operator $\mathcal{L}_\alpha$ contains the other p_0's. However, equation (4.22) is more convenient to handle than the expression (4.11), since it replaces the matrix X by a diagonal matrix at the price of introducing the additional term $\mathcal{L}_\alpha$. The momentum and kinetic energy operators can be obtained by substituting equations (4.20) and (4.22) into the general relations (4.13) and (4.14):

$$p_{i\mu} = \sum_{\alpha} a_{\alpha\mu}\pi_{i\alpha} + \hbar \sum_{\alpha\beta\sigma} \frac{a_{\alpha\mu}|\varepsilon_{\alpha\beta\sigma}|\xi_{i\beta}}{\varepsilon_{\sigma\gamma\delta}\xi_{i\beta}^2} (\lambda_\sigma - \Lambda_\sigma + \mathcal{L}_\sigma) , \tag{4.23}$$

$$2MT = \sum_{\nu i} \pi_{i\alpha}^2 + \hbar^2 \sum_{\substack{i\alpha\beta \\ \gamma\sigma\varkappa}} \frac{|\varepsilon_{\sigma\alpha\beta}||\varepsilon_{\varkappa\alpha\gamma}|\xi_{i\beta}\xi_{i\gamma}}{\sum_{\delta\nu} \varepsilon_{\alpha\delta\nu}\xi_{i\delta}^2} (\lambda_\sigma - \Lambda_\sigma + \mathcal{L}_\sigma) \frac{1}{\sum_{\delta\nu} \varepsilon_{\varkappa\delta\nu}\xi_{i\delta}^2} \cdot$$
$$\cdot (\lambda_\varkappa - \Lambda_\varkappa + \mathcal{L}_\varkappa) + 2\hbar \sum_{i\alpha\beta\sigma} |\varepsilon_{\alpha\beta\sigma}| \xi_{i\beta}\pi_{i\alpha} \frac{1}{\sum_{\gamma\delta} \varepsilon_{\sigma\alpha\delta}\xi_{i\gamma}^2} (\lambda_\sigma - \Lambda_\sigma + \mathcal{L}_\sigma) . \tag{4.24}$$

The kinetic energy can be written in a much simpler form as a semi-physical operator. Noting that $\mathcal{L}_\sigma$ is a semi-physical operator corresponding to the physical operator zero, we can discard it (*). All sums of the form $\sum_i \xi_{i\alpha}\xi_{i\beta}$

(*) Special precautions must be observed in the replacement of operators by their semiphysical equivalents. In particular, an operator equivalent semi-physically to zero cannot be automatically rejected if it is multiplied on the left by an operator with which it does not commute. In the case of equation (4.24), this causes no difficulty as the commutator $[(1/\sum_{i\gamma\delta} \varepsilon_{\sigma\lambda\delta}\xi_{i\gamma}^2)(\Lambda_\sigma - \lambda_\sigma), \mathcal{L}_\sigma]$ is semi-physically equivalent to zero. This follows from equation (4.22) and the fact that $\mathcal{L}_\sigma$ commutes with $p_{0\sigma}$.

can be replaced by $\sum_i \delta_{\alpha\beta}\xi_{i\beta}^2$. Then, defining an operator μ_σ by the relation:

$$\hbar\mu_\sigma = \sum_{i\alpha\beta} |\varepsilon_{\sigma\alpha\beta}| \xi_{i\alpha}\pi_{i\beta}\,, \tag{4.25}$$

we can write a semi-physical kinetic energy operator in the form:

$$2MT \simeq \sum_{i\alpha}\pi_{i\alpha}^2 + \hbar^2 \sum_\sigma \frac{\sum_{i\alpha\beta} |\varepsilon_{\sigma\alpha\beta}| \xi_{i\alpha}^2 (\Lambda_\sigma - \lambda_\sigma)^2}{\sum_{i\alpha\beta} \varepsilon_{\sigma\alpha\beta}\xi_{i\alpha}^2} + \hbar^2 \sum_\sigma \frac{2\mu_\sigma(\Lambda_\sigma - \lambda_\sigma)}{\sum_{\alpha\beta i} \varepsilon_{\alpha\alpha\beta}\xi_{i\alpha}^2}\,,$$

$$2MT \simeq \sum_{i\alpha}\pi_{i\alpha}^2 + \hbar^2 \left[\frac{\sum_i (\xi_{i\alpha}^2 - \xi_{i\beta}^2)}{\left\{\sum_i (\xi_{i\alpha}^2 - \xi_{i\beta}^2)\right\}^2} (\Lambda_\sigma - \lambda_\sigma)^2 + \right. \tag{4.26}$$

$$\left. + \frac{2\mu_\sigma(\Lambda_\sigma - \lambda_\sigma)}{\sum_i (\xi_{i\alpha}^2 - \xi_{i\beta}^2)} + \text{cyc. perm. of } \alpha\beta\sigma\,, \right]$$

where $\simeq$ means « equivalent semi-physically to ».

The rotational energy has exactly the form given by A. Bohr [2] including the expression for the moment of inertia. However, the coupling energy does not necessarily vanish.

One case for which the coupling term vanishes is for an anisotropic harmonic oscillator potential in which the intrinsic wave function corresponds to closed shells. The intrinsic part of the wave function consists of a polynomial in the ξ's multiplied by $\exp(-\sum_{i\alpha} k_\alpha \xi_{i\alpha}^2)$. Noting that the operators

$$\lambda_\sigma' = \exp\left[-\sum_{i\alpha} k_\alpha \xi_{i\alpha}^2\right]\lambda_\sigma \exp\left[\sum_{i\alpha} k_\alpha \xi_{i\alpha}^2\right] \text{ and } \mu_\sigma' = \exp\left[-\sum_{i\alpha} k_\alpha \xi_{i\alpha}^2\right]\mu_\sigma \exp\left[\sum_{i\alpha} k_\alpha \xi_{i\alpha}^2\right],$$

are equivalent semi-physically to λ_σ and μ_σ respectively we can replace the unprimed operators by the corresponding primed operators in the kinetic energy. The operators λ' and μ' operating on the harmonic oscillator wave function operate only upon the polynomial and not upon the exponential. Since they all consist of sums of single particle terms having the form $\xi_{i\alpha}\pi_{i\beta}$, the effect on the three dimensional oscillator wave function is to change the state of particle i by increasing by one the degree of the polynomial in $\xi_{i\alpha}$ and decreasing by one the degree of the polynomial in $\xi_{i\beta}$, i.e. to shift the particle i to another state in the *same shell*, having one quantum of energy less in one direction and one more in another. If all the states in the shell are filled, these operators all give zero, and there is no coupling between the intrinsic and collective motion.

5. – Extension to More General Transformations.

The general result obtained for the rotational case can be applied to other types of transformations than the one discussed in (4.3) ([12]), by using other forms for the relations (4.19) expressing the superfluous co-ordinates $x_{0\sigma}$ in terms of the ξ's. The general method can be applied to more general types of collective motion, such as vibrations. There will in general not be any invariance property associated with these motions; the potential therefore will not remain invariant under the transformation, and the collective co-ordinates will appear in the new potential. If the effect of changes in the collective co-ordinates from their equilibrium values is small, the potential can be expanded in a Taylor's series in the displacements of the co-ordinates from their equilibrium values. If the adiabatic approximation is valid and if all terms of higher order than quadratic can be neglected, a harmonic oscillator-type potential is obtained for the collective co-ordinates. This should be analogous to the « surface oscillators » discussed by BOHR and MOTTELSON ([2]) and by HILL and WHEELER ([13]), but the oscillator constants can be calculated from the intrinsic co-ordinate wave functions.

The more general co-ordinates can be introduced in a manner similar to that used for the rotational case. The necessary number of superfluous co-ordinates is introduced, and a transformation is defined to a new co-ordinate system, specified by collective co-ordinates ξ_0 which are equal in number to the superfluous co-ordinates. These collective co-ordinates are functions of the particle co-ordinates and of the superfluous co-ordinates. The functional relations defining the co-ordinates are generally of an implicit nature, making difficult the direct calculation of the transformation of the momenta. This transformation can be effected using infinitesimal transformation operators Ω_σ and ω_σ, analogous to the operators Λ_σ and λ_σ for the rotational case. The operators Ω_σ are operators of infinitesimal transformation in the space of the particles x_i, corresponding to the same type of transformation as that generating the collective motion. These operators have the form

$$\Omega_\sigma = \sum_{i\mu} b_{\sigma\mu i} p_{i\mu} , \tag{5.1}$$

where the coefficients $b_{\sigma\mu i}$ are functions of the x_i and x_τ, but not of the momenta. By definition, the operators Ω_σ commute with the intrinsic co-ordin

([12]) The use of other transformations has been investigated by A. BOHR: Private communication.

([13]) D. L. HILL and J. A. WHEELER: *Phys. Rev.*, **89**, 1102 (1953).

ates ξ_α, which are defined as the particle co-ordinates relative to the co-ordinate axes defined by ξ_0: The operators Ω_σ, generating infinitesimal transformations of the co-ordinate axes, operate only on the collective co-ordinates ξ_0, and are in general momenta canonically conjugate to some function of the co-ordinates ξ_0. If an operator ω_σ can be defined, representing infinitesimal transformations in the ξ_i space, and having the property

$$i[\omega_\sigma, x_{i\mu}] = b_{\sigma\mu i} , \tag{5.2}$$

then the momenta $p_{i\mu}$ can be expressed in terms of the co-ordinates $\xi_{i\lambda}$ and the operators ω_σ and Ω_σ in a manner identical to the rotational case.

The expression for the intrinsic momenta $\pi_{i\alpha}$ is first written

$$\pi_{i\alpha} = \sum_{j\mu} (\partial x_{j\mu}/\partial \xi_{i\alpha}) p_{j\mu} + \sum_{\sigma} (\partial x_{0\sigma}/\partial \xi_{i\alpha}) p_{0\sigma} . \tag{5.3}$$

This expression can be solved for the momenta p_j, if the $p_{0\sigma}$ are known. The latter are obtained by writing an expression for the operators ω_σ. Since these are infinitesimal transformation operators, they are linear combinations of the $\pi_{i\alpha}$, with coefficients depending upon the $\xi_{i\lambda}$, and are therefore given directly from equation (5.3) by the relation:

$$\omega_\sigma = i \sum_{j\mu} [\omega_\sigma, x_{j\mu}] p_{j\mu} + i \sum_{\nu} [\omega_\sigma, x_{0\nu}] p_{0\nu} . \tag{5.4}$$

As in the rotational case, define a matrix $(Y)_{\sigma\nu}$ by the relation

$$Y_{\sigma\nu} = i[\omega_\sigma, x_{0\nu}] . \tag{5.5}$$

The first term on the right hand side of equation (5.4) is just Ω_σ, by the definitions (5.2) and (5.1). Equation (5.4) can therefore be solved for the momenta $p_{0\nu}$, giving

$$p_{0\nu} = (Y)^{-1}_{\nu\sigma} (\omega_\sigma - \Omega_\sigma) . \tag{5.6}$$

From this point the procedure is exactly analogous to the rotational case to obtain the kinetic energy.

The occurence in the rotational Hamiltonian (4.26) of the operator μ_σ, an operator corresponding to an infinitesimal deformation, and of an effective moment of inertia depending upon deformation parameters suggests that a more complete description would be given including collective co-ordinates corresponding to deformations and leading to collective vibrational motion. This has been done by BOHR and MOTTELSON [1], and is also suggested by

the form of the expression (4.19) for $x_{0\sigma}$, which is that of three components of a second order tensor, and not of a vector as indicated by the notation. One might try the addition of two additional co-ordinates corresponding to the remaining two components of the quadrupole tensor and analogous to the deformation parameters β and γ of BOHR and MOTTELSON [1] and perhaps a sixth co-ordinate corresponding to volume oscillations.

* * *

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RIASSUNTO (*)

Si espone un metodo, basato sull'introduzione di coordinate superflue, atto a trattare il problema della struttura nucleare considerata contemporaneamente nei suoi aspetti a particelle singole o collettivo. L'applicazione al caso del centro di massa permette una esposizione semplice del metodo e definisce rigorosamente le condizioni per la validità delle ipotesi riguardanti il centro di massa prese a base del consueto modello a shell. L'applicazione al caso rotativo dà risultati simili a quelli ottenuti da A. BOHR servendosi di un modello idrodinamico e da INGLIS servendosi di un classico modello a shell. Il trattamento che presentiamo è rigoroso eccetto che per l'ipotesi fondamentale del modello a shell, la sostituzione al potenziale fra particelle di un potenziale monoparticellare che può non esser dotato di simmetria sferica.

(*) *Traduzione a cura della Redazione.*

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Condizioni al contorno nella teoria quantistica della dispersione.

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Aosta

(ricevuto il 20 Luglio 1955)

Riassunto. — Si presenta una nuova trattazione della teoria quantistica della dispersione che costituisce una approssimazione differente da quella originale di Dirac e di Heitler. Mediante una generalizzazione dell'operatore risolvente si trova, sia nel caso stazionario che in quello non stazionario, una soluzione dipendente da un parametro arbitrario c, la quale è più generale dell'usuale soluzione perturbativa e tende a questa nel limite $c \to 0$. Tale soluzione è usata per determinare la variazione dello spostamento e della larghezza della riga di un sistema quando ad esso è applicata una perturbazione H'. Infatti, introducendo la dipendenza di c dal quanto azimutale l, c_l può esprimersi in funzione dello sfasamento $\delta_l^{(0)}$ del sistema imperturbato e della perturbazione H'; inoltre c_l è univocamente collegato con lo sfasamento δ_l del sistema perturbato. La parte reale di c_l dà lo spostamento del livello di risonanza, mentre la parte immaginaria dà la larghezza della riga corrispondente allo stato l. Tali grandezze sono in tal modo connesse, attraverso $\delta_l^{(0)}$, con le condizioni al contorno del sistema imperturbato.

Introduzione.

La trattazione della teoria della dispersione che sarà presentata in questo lavoro si fonda su una generalizzazione dell'operatore risolvente $R(\lambda)$ introdotto nella meccanica quantistica da Kato (1) e Schönberg (2). Mediante l'operatore risolvente generalizzato $R'(\lambda)$, introdotto nel § 1, si definiranno (§ 2) una matrice S_0' e una matrice $V_1'(t)$ le quali sono soluzioni, rispettivamente

(1) T. Kato: *Progr. Theor. Phys.*, **4**, 514 (1949); **5**, 95, 207 (1950).
(2) M. Schönberg: *Nuovo Cimento*, **8**, 651 (1951).

stazionaria e non stazionaria, più generali della equazione del moto: tali soluzioni dipendono infatti da un parametro c a priori arbitrario e sono tali che coincidono rispettivamente con la matrice S di Heisenberg e con la matrice $V_+(t)$, quale è definita da SCHÖNBERG [2], nel limite $c \to 0$.

Queste soluzioni, come si mostrerà nel § **3**, generalizzano le soluzioni usuali in maniera che possano essere descritti i fenomeni di risonanza. Lo spostamento e la larghezza della riga sono dati rispettivamente dalla parte reale e immaginaria del parametro c. Assumendo la dipendenza di c dal quanto azimutale l, esso può univocamente collegarsi con lo sfasamento δ_l. Il valore di c_l dipende dalle condizioni al contorno del particolare problema da trattare. Esso può altresì determinarsi dalla condizione che nel limite in cui la perturbazione H' è nulla, lo sfasamento δ_l coincida con lo sfasamento del sistema imperturbato (§ **4**).

I c_l sono così determinati da H' e dagli sfasamenti $\delta_l^{(0)}$ del sistema imperturbato. Il formalismo consente in tal modo di calcolare la modificazione dello spostamento e della larghezza della riga dovuta all'esistenza della perturbazione H'.

Il cosidetto termine di « damping », che riveste una parte fondamentale nella trattazione originale di DIRAC [3], in quella di HEITLER [4] e degli autori successivi [5], non gioca alcuna parte nella presente trattazione, nonostante che sia in essa implicito che per ottenere una larghezza della riga non nulla e positiva sia necessario considerare onde diffuse divergenti.

Come esempio si tratta brevemente (§ **4**) il caso particolare di un diffusore delimitato da una superficie sferica di raggio r_0. In quest'esempio si mostra che nel limite $r_0 \to \infty$, cioè nel caso in cui il sistema imperturbato sia costituito da un diffusore rappresentato da un potenziale a lungo raggio d'azione, la presente approssimazione è equivalente alla trattazione usuale.

1. – L'operatore risolvente generalizzato $R'(\lambda)$.

L'operatore risolvente ordinario è definito dalla relazione

$$R(\lambda) = \frac{1}{H - \lambda}, \tag{1}$$

[3] P. A. M. DIRAC: *The Principles of Quantum Mechanics* (Oxford, 1947), p. 199.

[4] W. HEITLER: *Quantum Theory of Radiation* (Oxford, 1954), p. 163; W. HEITLER e H. W. PENG: *Proc. Cam. Phil. Soc.*, **38**, 296 (1942); W. HEITLER e S. T. MA: *Proc. Roy. Irish Acad.*, A **52**, 109 (1949).

[5] W. PAULI: *Meson Theory of Nuclear Forces* (New York, 1946), p. 41; J. HAMILTON: *Proc. Phys. Soc.*, **62**, 12 (1949); N. FUKUDA e T. MIYAZIMA: *Progr. Theor. Phys.*, **5**, 849 (1950); M. SCHÖNBERG: *Nuovo Cimento*, **8**, 403, 817 (1951).

ove H è l'hamiltoniana del sistema in considerazione e λ un parametro reale o complesso non appartenente allo spettro di H. Una generalizzazione dell'operatore (1) è ottenuta considerando una matrice $R'(\lambda)$ i cui elementi $\langle m|R'(\lambda)|m'\rangle$ per definizione soddisfino alle equazioni

$$(2) \qquad (\lambda - E_{0m})\langle m|R'(\lambda)|m'\rangle =$$

$$= \sum_n \langle m|H'|n\rangle\langle n|R'(\lambda)|m'\rangle + c_{m'}\,\delta_{mm'}\langle m|R'(\lambda)|m'\rangle - \delta_{mm'}\,, \quad (m = 1, 2, \ldots)$$

nelle quali E_{0m} è un autovalore dell'hamiltoniana imperturbata, H' è la perturbazione e λ e $c_{m'}$ sono due parametri in generale complessi. Per semplicità ci limitiamo in questo numero a considerare solo il caso di uno spettro discreto dell'hamiltoniana imperturbata.

Se è $c_{m'} = 0$, si ottiene dalle (2) la definizione (1) dell'operatore usuale $R(\lambda)$.

Onde meglio illustrare la relazione tra l'operatore generalizzato $R'(\lambda)$ così definito e l'operatore risolvente ordinario $R(\lambda)$, cominciamo con il mostrare che se esiste una soluzione delle (2) tale che soddisfi alla condizione

$$(3) \qquad \langle m'|R'(\lambda)|m'\rangle = \frac{1}{c_{m'}}\,,$$

il parametro λ deve essere uguale all'autovalore $E_{m'}$ dell'hamiltoniana $H_0 + H'$. È infatti facile dimostrare che condizione necessaria e sufficiente perchè le (2) abbiano una soluzione tale che la (3) sia soddisfatta, è che λ soddisfi all'equazione secolare.

La necessità della condizione segue subito dal fatto che se gli elementi di matrice $\langle m|R'(\lambda)|m'\rangle$ soddisfano alle (2), ed è soddisfatta la (3), tali elementi soddisfano altresì al sistema lineare:

$$(4) \quad \sum_n (E_{0m}\delta_{mn} + \langle m|H'|n\rangle)\langle n|R'(\lambda)|m'\rangle - \lambda\langle m|R'(\lambda)|m'\rangle = 0, \quad (m = 1.2, \ldots)$$

per cui λ deve soddisfare all'equazione secolare:

$$(5) \qquad \|E_{0m}\delta_{mn} + \langle m|H'|n\rangle - \lambda\delta_{mn}\| = 0\,.$$

Viceversa se λ soddisfa all'equazione secolare, ed è perciò uguale ad un autovalore $E_{m'}$, esiste una matrice S i cui elementi soddisfano al sistema lineare:

$$(6) \quad \sum_m \langle m'|S|m\rangle(E_{0m}\delta_{mn} + \langle m|H'|n\rangle) - E_{m'}\langle m'|S|n\rangle = 0\,, \qquad (n = 1, 2, \ldots).$$

Moltiplicando il primo e il secondo membro della (2) per $\langle m'|S|m\rangle$, som-

mando in m, e tenendo conto delle (6), si riottiene subito la (3). Le radici in λ della (3) sono perciò autovalori dell'hamiltoniana $H = H_0 + H'$ e la (3) coincide con l'equazione secolare.

Quando $c_{m'}$ è nullo l'operatore risolvente generalizzato è uguale all'operatore risolvente ordinario e la (3) contiene quindi il risultato noto secondo il quale gli autovalori sono poli del risolvente ordinario.

Otteniamo ora uno sviluppo in serie per il risolvente generalizzato. Per questo consideriamo $R'(\lambda)$ come una funzione analitica di un parametro fittizio z e l'hamiltoniana sia $H_0 + zH'$. Lo sviluppo di $R'(\lambda)$ è ottenuto dalla (2) sviluppando $R'(\lambda)$ in serie di potenze di z, uguagliando i coefficienti delle potenze uguali di z e ponendo poi $z = 1$ nel risultato finale. Le espressioni degli elementi di matrice nei diversi ordini dello sviluppo sono:

$$\langle m | R'^{(0)} | m' \rangle = - \frac{\delta_{mm'}}{\lambda - E_{0m} - c_{m'}\delta_{mm'}}, \tag{7a}$$

$$\langle m | R'^{(n)} | m' \rangle = \tag{7b}$$

$$= - \frac{1}{(\lambda - E_{0m} - c_{m'}\delta_{mm'})(\lambda - E_{0m'} - c_{m'})} \sum_{k_1, \ldots, k_n} \frac{\langle m = k_0 | H' | k_1 \rangle \ldots \langle k_n | H' | m' \rangle}{\prod_1^{n-1}{}_i (\lambda - E_{0k_i} - c_{m'}\delta_{k_i m'})}.$$

Poichè il parametro $c_{m'}$ può essere funzione della costante di accoppiamento, il parametro fittizio z precedentemente introdotto non può in generale identificarsi con tale costante dato che nel procedimento fatto si è assunto $c_{m'}$ indipendente da z. Lo sviluppo in serie che si ottiene dalle (7) non è quindi in generale uno sviluppo in serie di potenze della costante di accoppiamento.

Lo sviluppo (7) può porsi in una forma più compatta ed elegante notando che esso può essere ottenuto mediante una serie di Liouville-Neumann.

Consideriamo per questo scopo l'equazione dell'operatore risolvente ordinario:

$$(H_0 + H' - \lambda) R(\lambda) = 1, \tag{8}$$

il cui sviluppo in serie di Liouville-Neumann è:

$$R(\lambda) = \sum_0^\infty{}_n (-1)^n [R_0(\lambda) H']^n R_0(\lambda). \tag{9}$$

Notiamo che se nella (8) al posto del parametro λ si sostituisce una matrice $\Lambda^{(m')}$ i cui elementi siano

$$\langle m | \Lambda^{(m')} | n \rangle = (\lambda - c_{m'}\delta_{mm'})\delta_{mn}, \tag{10}$$

ogni qualvolta si passi nella (8) agli elementi di matrice relativi allo stato iniziale m' (m' essendo così connesso con la definizione (10) di $\Lambda^{(m')}$), si ottengono le relazioni (2) cui soddisfano gli elementi di matrice del risolvente generalizzato $R'(\lambda)$. Pertanto si ottengono gli elementi di matrice $\langle m|R'(\lambda)|m'\rangle$ della colonna m' della matrice $R'(\lambda)$ semplicemente sostituendo nel risolvente ordinario al parametro λ la matrice $\Lambda^{(m')}$ sopra definita. Si ha così dalla (9) lo sviluppo in serie

$$\langle m|R'(\lambda)|m'\rangle = \langle m|\sum_0^\infty{}_n (-1)^n [R_0\{\Lambda^{(m')}(\lambda)\}H']^n R_0\{\Lambda^{(m')}(\lambda)\}|m'\rangle\,, \tag{11}$$

il quale coincide con lo sviluppo che si ottiene dalle (6), come si può facilmente verificare.

La matrice $R'(\lambda)$ differisce quindi dalla matrice $R(\lambda)$ per il fatto che, per ottenere gli elementi di una colonna m' si è sommato all'autovalore $E_{0m'}$ (e soltanto a questo autovalore) la quantità arbitraria $c_{m'}$.

2. – Le matrici generalizzate S' e $V'_+(t)$.

Usiamo ora l'operatore risolvente generalizzato $R'(\lambda)$ precedentemente definito per ottenere delle soluzioni più generali dell'equazione del moto secondo lo stesso formalismo dell'operatore risolvente ordinario.

Introduciamo dunque una matrice S' i cui elementi siano ottenuti dalla relazione

$$\langle m|S'|m'\rangle = -\frac{1}{2\pi i}\oint \langle m|R'(\lambda)|m'\rangle\,\mathrm{d}\lambda\,, \tag{12}$$

nella quale il cammino chiuso di integrazione, percorso in senso antiorario nel piano complesso di λ, contiene il polo $\lambda = E_{0m'} + c_{m'}$ e non gli altri poli dei secondi membri delle (7).

La matrice S' così definita diagonalizza l'hamiltoniana $H_0 + H'$ secondo i suoi autovalori reali. Si ha infatti:

$$\begin{aligned}\sum \langle m|H|n\rangle\langle n|S'|m'\rangle &= \\ &= E_{0m}\langle m|S'|m'\rangle - \frac{1}{2\pi i}\oint \langle m|(H-H_0)R'(\lambda|(m'\rangle\,\mathrm{d}\lambda = \\ &= E_{0m}\langle m|S'|m'\rangle - \frac{1}{2\pi i}\oint (\lambda - E_{0m} - c_{m'}\delta_{mm'})\langle m|R'(\lambda)|m'\rangle\,\mathrm{d}\lambda\,.\end{aligned} \tag{13}$$

Poichè è $\langle m'|S'|m'\rangle = 1$, si ottiene:

$$(14)\qquad E_{m'} = \sum_n \langle m'|H|n\rangle\langle n|S'|m'\rangle =$$

$$= E_{0m'} - \frac{1}{2\pi i}\oint (\lambda - E_{0m'} - c_{m'})\langle m'|R'(\lambda)|m'\rangle \mathrm{d}\lambda =$$

$$= E_{0m'} + \langle m'|H'|m'\rangle + \sum_{n\neq m'} \frac{\langle m'|H'|n\rangle\langle n|H'|m'\rangle}{E_{0m'} - E_{0n}} + \dots .$$

Si ritrova così lo sviluppo usuale per gli autovalori perturbati. La matrice S' così ottenuta diviene uguale alla matrice S, definita con il risolvente ordinario, nel limite $c_{m'} \to 0$.

Mediante l'estensione al risolvente generalizzato $R'(\lambda)$ della definizione di SCHÖNBERG [2] dell'operatore del movimento $V_+(t)$ per mezzo del risolvente ordinario $R(\lambda)$, si ottiene un più generale operatore del movimento $V'_+(t)$, che si riduce al precedente per $c_{m'} = 0$:

$$(15)\qquad \langle m|V'_+(t)|m'\rangle = \frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \exp[-i\lambda t]\langle m|R'(\lambda)|m'\rangle \mathrm{d}\lambda , \qquad (t>0)$$

ove ε è un infinitesimo positivo e il cammino di integrazione è un semicerchio di raggio infinito, con centro nell'origine e giacente nel piano complesso inferiore di λ. La parte immaginaria del parametro $c_{m'}$ è supposta negativa: quest'ipotesi sarà giustificata nel seguito quando si mostrerà che tale parte, mutata di segno, è la larghezza della riga.

Gli elementi di matrice $\langle m|V'_+(t)|m'\rangle$ con $m \neq m'$ soddisfano all'equazione del movimento:

$$(16)\quad \langle m|HV'_+(t)|m'\rangle = \frac{1}{2\pi i}\int_{\infty-i\varepsilon}^{+\infty+i\varepsilon} \exp[-i\lambda t]\langle m|HR'(\lambda)|m'\rangle \mathrm{d}\lambda =$$

$$= \frac{1}{2\pi i}\int_{\infty-i\varepsilon}^{-\infty-i\varepsilon} \lambda \exp[-i\lambda t]\langle m|R'(\lambda)|m'\rangle \mathrm{d}\lambda = i\left\langle m\left|\frac{\mathrm{d}V'_+(t)}{\mathrm{d}t}\right|m'\right\rangle , \quad (t>0).$$

Si ha inoltre:

$$(17)\quad \langle m'|V'_+(t)\ m'\rangle = \frac{1}{2\pi i}\int_{\infty-i\varepsilon}^{+\infty+i\varepsilon} \exp[-i\lambda t]\langle m'|R'(\lambda)|m'\rangle \mathrm{d}\lambda = \exp[-it(E_{0m'} + c_{m'})] .$$

La matrice $\langle m'|V'_+(t)|m'\rangle$ diagonalizza l'hamiltoniana:

$$(18)\quad \langle m'|HV'_+(t)|m'\rangle = \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty+i\varepsilon}\exp[-i\lambda t]\langle m'|HR'(\lambda)|m'\rangle\,d\lambda =$$

$$= \frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{-\infty-i\varepsilon}\exp[-i\lambda t](\lambda - c_{m'})\langle m'|R'(\lambda)|m'\rangle\,d\lambda =$$

$$= \exp[-it(E_{0m'}+c_{m'})]\left(E_{0m'}+\langle m'|H'|m'\rangle + \sum_{n\neq m'}\frac{\langle m'|H'|n\rangle\langle n|H'|m'}{E_{0m'}-E_{0n}} - \ldots\right).$$

Mediante la (11) si ottiene immediatamente dalla (12) o dalla (15) uno sviluppo in serie espresso in forma compatta per la matrice S' o la matrice $V'_+(t)$.

3. – Teoria della risonanza con le matrici S' e $V'_+(t)$.

Vogliamo ora brevemente mostrare che le matrici S' e $V'_+(t)$ descrivono i fenomeni di risonanza. In questo numero non determiniamo il parametro c, ciò che sarà fatto nel numero seguente, ma ci limitiamo a notare che se c è formato da una parte reale a e da una parte immaginaria negativa $-ib$ $(b>0)$, si ottengono formule che hanno la forma generale della teoria della risonanza.

Pertanto consideriamo che l'hamiltoniana imperturbata abbia uno spettro misto di autovalori; siano $E_0 = H_d(\alpha_0) + W(p_0)$ l'energia dello stato iniziale del sistema imperturbato formato dal diffusore nello stato α_0 e dalla particella di energia cinetica $W(p_0)$, $E(p) = H_d(\alpha') + W(p)$ l'energia di questo sistema nello stato finale, E_k gli autovalori dell'hamiltoniana imperturbata relativi agli stati legati che si suppongono indipendenti e ortogonali rispetto agli stati del continuo.

Gli elementi di matrice di $A^{(E_0)}$ sono nel presente caso

$$(19a)\quad \langle \boldsymbol{p}_1\alpha_1|A^{(E_0)}|\boldsymbol{p}_2\alpha_2\rangle = [\lambda - c(E_0)\gamma(E(p_1)-E_0)]\delta(\boldsymbol{p}_1-\boldsymbol{p}_2)\delta_{\alpha_1\alpha_2},$$

$$(19b)\quad \langle k_1|A^{(E_0)}|k_2\rangle = [\lambda - c(E_0)\gamma(E_{k_1}-E_0)]\delta_{k_1k_2},$$

$$(19c)\quad \langle k|A^{(E_0)}|\boldsymbol{p}\alpha\rangle = \langle \boldsymbol{p}\alpha|A^{(E_0)}|k\rangle = 0;$$

ove $\gamma(y-y')$ è una funzione della variabile continua y' (tale funzione tiene il posto della $\delta_{mm'}$ dell'elemento di matrice (10)) così definita:

$$(20)\quad \begin{cases} \gamma(y-y') = 0 & \text{per } y'\neq y, \\ \gamma(y-y') = 1 & \text{per } y' = y. \end{cases}$$

In tale modo gli elementi di matrice di $R'(\lambda)$ relativi allo stato iniziale di energia E_0 sono ottenuti dagli elementi di matrice del risolvente ordinario sostituendo $A^{(E_0)}$ a λ, ovvero sommando ad E_0 (e solo a questo autovalore) il parametro $c(E_0)$, in accordo con il procedimento del § 1.

a) *Caso stazionario.* Con le usuali assunzioni si ottengono immediatamente dalla (7*b*) e dalla (12) i seguenti elementi di matrice del primo e del secondo ordine, che hanno la forma caratteristica della teoria della risonanza:

$$\langle m | S'^{(1)} | \boldsymbol{p}_0 \alpha_0 \rangle = \frac{\langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{E_0 - E_{0m} + a - ib}, \qquad (E_{0m} \neq E_0), \tag{21}$$

$$\langle \boldsymbol{p}\alpha' | S'^{(2)} | \boldsymbol{p}_0 \alpha_0 \rangle = \frac{\langle \boldsymbol{p}\alpha' | H' | m \rangle \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{\{E_0 + a - ib - E(p) - (a - ib)\gamma(E(p) - E_0)\}(E_0 + a - ib - E_{0m})} \qquad (E_{0m} \neq E_0). \tag{22}$$

In queste e nelle seguenti E_0 è supposto molto prossimo a E_{0m}. Dalla relazione

$$u(r, \theta, \varphi) = \int \exp[-i\boldsymbol{p}\cdot\boldsymbol{x}] \langle \boldsymbol{p}\alpha' | S'^{(2)} | \boldsymbol{p}_0 \alpha_0 \rangle \, \mathrm{d}^3\boldsymbol{p}, \tag{23}$$

si ottiene, usando, secondo il metodo noto, il teorema dei residui e con opportuna scelta del cammino di integrazione per ottenere onde diffuse divergenti, nel limite non relativistico e per $r \to \infty$:

$$u(r, \theta, \varphi) = 2\pi\mu \frac{\exp[-ipr]}{r} \frac{\langle \boldsymbol{p}\alpha' | H' | m \rangle \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{E_0 - E_{0m} + a - ib}, \qquad (E_{0m} \neq E_0). \tag{24}$$

b) *Caso non stazionario.* Dalla (7*b*) e dalla (15) si ottiene:

$$\langle m | V'^{(1)}_+(t) | \boldsymbol{p}_0 \alpha_0 \rangle = \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle \cdot \frac{\exp[-it(E_{0m} - E_0)] - \exp[-it(a - ib)]}{E_0 - E_{0m} + a - ib} \exp[-itE_0], \qquad (E_{0m} \neq E_0), \tag{25}$$

$$\begin{aligned} \langle \boldsymbol{p}\alpha' | V'^{(2)}_+(t) | \boldsymbol{p}_0 \alpha_0 \rangle = & \\ - \frac{\langle \boldsymbol{p}\alpha' | H' | m \rangle \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{\{E_0 + a - ib - E(p) - (a - ib)\gamma(E(p) - E_0)\}(E_0 + a - ib - E_{0m})} & \cdot \\ \cdot \exp[-it(E_0 + a - ib)] + & \\ + \frac{\langle \boldsymbol{p}\alpha' | H' | m \rangle \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{\{E(p) + (a - ib)\gamma(E(p) - E_0) - E_0 - a + ib\}\{E(p) - E_{0m} + (a - ib)\gamma(E(p) - E_0)\}} & \\ \cdot \exp[-it\{E(p) + (a - ib)\gamma(E(p) - E_0)\}] + & \\ (E_{0m} - E_0 - a + ib)\{E_{0m} - E(p) - (a - ib)\gamma(E(p) - E_0)\} \exp[-itE_{0m}], \qquad (E_{0m} \neq \end{aligned} \tag{26}$$

Nel limite $t \to +\infty$ la (25) assume la forma solita:

$$(27)\quad \langle m | V_+'^{(1)}(t) | \boldsymbol{p}_0 \alpha_0 \rangle = \frac{\langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle}{E_0 - E_{0m} + a - ib} \exp[-itE_0], \qquad (t \to +\infty;\ E_{0m} \neq E_0).$$

Per ottenere il limite $t \to +\infty$ della (26) occorre introdurre, onde avere risultati consistenti, la prescrizione di sommare allo stato legato E_{0m} l'infinitesimo immaginario $-i\eta$ $(\eta > 0)$; si ha allora:

$$(28)\quad \langle \boldsymbol{p}\alpha' | V_+'^{(2)}(t) | \boldsymbol{p}_0 \alpha_0 \rangle = \frac{\langle \boldsymbol{p}\alpha' | H' | m \rangle \langle m | H' | \boldsymbol{p}_0 \alpha_0 \rangle \exp[-it\{E(p) + (a - ib)\gamma(E(p) - E_0)\}]}{\{E(p) + (a - ib)\gamma(E(p) - E_0) - E_0 - a + ib\}\{E(p) - E_{0m} + (a - ib)\gamma(E(p) - E_0)\}},$$
$$(t \to +\infty;\ E_{0m} \neq E_0).$$

Il coefficiente di diffusione è subito ottenuto integrando la precedente con il teorema dei residui rispetto all'energia $E(p)$ degli stati finali (ma trascurando, come nella trattazione corrente, la dipendenza da $E(p)$ dell'elemento di matrice) con un cammino di integrazione dato da un semicerchio giacente nel piano complesso inferiore di $E(p)$, con centro nell'origine e con diametro infinito dato dalla retta $y = i\sigma$ (con $\sigma > 0$); indi moltiplicando il quadrato del modulo per $W(p_0) \cdot W(p) \cdot p/p_0$.

4. – La determinazione dello spostamento e della larghezza della riga.

La presente teoria deve essere completata in maniera da determinare il parametro a priori arbitrario $c(E_0)$. Per questo è conveniente suddividere la funzione d'onda nelle componenti del quanto azimutale l e approfittare dell'arbitrarietà di $c(E_0)$ per assumere una dipendenza di $c(E_0)$ da l.

Scriviamo quindi, con ovvie notazioni e limitandoci al caso stazionario di urto elastico con quanto magnetico m dell'onda piana incidente nullo:

$$(29)\qquad \psi(\boldsymbol{r}, \boldsymbol{p}_0) = \sum_l \int \langle pl | S_l' | \boldsymbol{p}_0 \rangle \sqrt{\frac{2l+1}{4\pi}} \frac{u_l(pr)}{r} P_l(\cos\theta)\, \mathrm{d}^3\boldsymbol{p},$$

ove l'indice l apposto ad S' indica che questa matrice dipende da $c_l(E_0)$.

I $c_l(E_0)$ possono esprimersi facilmente in funzione degli sfasamenti $\delta_l = 1 - \eta_l$. Basta infatti uguagliare l'espressione asintotica dell'onda diffusa che si ottiene con la (29) alla ben nota espressione [6] coinvolgente gli sfasa-

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[6] Cfr. per es.: M. Blatt e V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), p. 320.

menti. Se, per esempio, si tiene conto del solo termine del secondo ordine della matrice S_l', si ha:

$$(30) \qquad \delta_l(E_0) = -8\pi^2 i p_0 W(p_0) \frac{|\langle p_0 e | H' | m \rangle|^2}{E_0 - E_{0m} + c_l(E_0)} .$$

Lo sfasamento δ_l sarà formato dalla somma di una parte $\delta_l^{(0)}$ relativa al sistema imperturbato e da una parte δ_l' che tenderà a zero con l'annullarsi della perturbazione H'. Determiniamo $(c_l(E_0))_{E_0=E_{0m}}$ dalla condizione che il secondo membro della (30), quando l'energia E_0 è uguale all'energia di risonanza $E_{0m}^{(7)}$, debba tendere a $\delta_l^{(0)}$ nel limite $H' \to 0$. Si ottiene allora immediatamente:

$$(31) \quad (c_l(E_0))_{E_0=E_{0m}} = -8\pi^2 i \left(p_0 W(p_0) \frac{|\langle p_0 l | H' | m \rangle|^2}{\delta_l^{(0)}(E_0)} \right)_{E_0=E_{0m}}, \qquad (\delta_l^{(0)} \neq 0) .$$

La larghezza e lo spostamento della riga sono così determinati dallo sfasamento del sistema imperturbato e dalla perturbazione H'. Tali grandezze dipendono quindi, attraverso $\delta_l^{(0)}$, dalle condizioni al contorno del sistema imperturbato. Esse sono in generale differenti dalle analoghe grandezze ottenute con l'approssimazione usuale della teoria della dispersione.

Onde meglio illustrare la presente trattazione della teoria della dispersione, consideriamo, come semplice esempio, il caso particolare di un diffusore il quale sia esattamente delimitato da una superficie sferica di raggio r_0 (per esempio la superficie nucleare). I $(c_l(E_0))_{E_0=E_{0m}}$ possono determinarsi direttamente dalla condizione che la derivante logaritmica $(f_{\text{est},l})_{E_0=E_{0m}}$ della l-esima componente della funzione $r\psi$, quando l'energia E_0 della particella incidente è uguale all'energia E_{0m} di risonanza (7), sia uguale, sulla superficie che delimita il diffusore, alla derivata logaritmica $f_{\text{int},l}$ della funzione $r\psi_m$ relativa allo stato intermedio legato di energia E_{0m} entro il diffusore. Gli sfasamenti sono poi determinati mediante la (30). Nel caso non stazionario la stessa condizione precedente può essere usata per determinare $(c_l(E_0))_{E_0=E_{0m}}$ nel limite $t \to +\infty$, quando si è raggiunta la condizione di equilibrio dinamico.

Tenendo conto dei soli termini di ordine zero e di ordine due della matrice S_l' la nostra condizione si scrive:

$$(32) \quad \begin{pmatrix} (-i)^l \left(\frac{\mathrm{d}u_l}{\mathrm{d}r}\right)_{r=r_0} c_l(E_0) + 4\pi^2 i(2l+1)\, \tfrac{1}{2} p_0 W(p_0) \langle p_0 l | H' | ml \rangle \langle ml | H' | \boldsymbol{p}_0 \left(\frac{\mathrm{d}}{\mathrm{d}r} r h_l^{(2)}\right)_{r=r_0} \\ (-i)^l (u_l)_{r=r_0} c_l(E_0) + 4\pi^2 i(2l+1)\, \tfrac{1}{2} p_0 W(p_0) \langle p_0 l | H' | ml \rangle \langle ml | H' | \boldsymbol{p}_0 \rangle (r h_l^{(2)})_{r=r_0} \end{pmatrix} = f_{\text{in}}$$

$$E_0 = E_{0}$$

(7) A rigore E_0 non può porsi uguale a E_{0m} negli elementi di matrice ottenuti di S_l' o $V_{+l}'(t)$, poichè in tal caso, come appare dalla (7*b*) e dalle (12) o (15), il termine di secondo ordine (e quelli di ordine superiore a questo) coinvolgente lo stato E_{0m} è nullo. Tuttavia E_0 può considerarsi infinitamente vicino a E_{0m}.

ove $h_l^{(2)}(p_0 r)$ è la funzione sferica di Hankel soddisfacente alla condizione di irraggiamento centrifugo. Con semplice calcolo, indicando con v_l e w_l rispettivamente la parte reale e immaginaria di $r h_l^{(2)}(p_0 r)$ nel punto $r = r_0$, e con v_l' e w_l' le rispettive derivate in questo stesso punto, si ottiene:

$$(32)\quad (c_l(E_0))_{E_0=E_{0m}} = a_l - ib_l =$$

$$= -4\pi^2 \left[p_0 W(p_0) \quad p_0 l \quad H' \quad ml \quad :^2 \left(i - \frac{w_l'(p_0 r_0) - f_{\mathrm{int},l} w_l(p_0 r_0)}{v_l'(p_0 r_0) - f_{\mathrm{int},l} v_l(p_0 r_0)} \right) \right]_{E_0=E_{0m}} .$$

Se $f_{\mathrm{int},l}$ è reale, la larghezza della riga coincide con l'espressione usuale. Inoltre, come si può facilmente vedere, si riottiene il risultato della trattazione usuale nel limite $r_0 \to \infty$. Per questo basta osservare che, se si assume, come d'uso in meccanica quantistica [8,9], che in questo limite le funzioni $\cos kr_0$, $\operatorname{sen} kr_0$ (k reale, diverso da zero), siano nulle, il secondo termine entro la parentesi dell'ultimo membro della (32) tende a 1 per ogni l.

Pertanto nel precedente esempio la presente approssimazione della teoria della dispersione è equivalente alla trattazione di Dirac nel limite di un diffusore rappresentato da un potenziale a lungo raggio d'azione.

* * *

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(8) M. Hardy: *Divergent Series* (Oxford, 1949), p. 12.
(9) P. A. M. Dirac: cfr. op. cit., p. 95.

SUMMARY (*)

A quantum elaboration of quantum theory of dispersion is exposed, which is an approximation different of the one of Dirac and Heitler. Through a generalization of the resolving operator in the standing case as well as in the non standing, a solution is found dependent on an arbitrary parameter c, which is more general than the usual

(*) *Editor's Translation.*

perturbative solution and tends towards it in the limit $c \to 0$. This solution is used for the determination of the variation of the shift and the width of the lines of a system when the latter is subjected to a perturbation H'. In fact, introducing the dependence of c on the azimuthal quantum l, c_l may be expressed as a function of the phase shift $\delta_l^{(0)}$ of the unperturbed system and of the perturbation H'; c_l is moreover univocally connected with the phase shift δ_l the perturbed system. The real part of c_l gives the shift of the resonance level, whereas the imaginary part gives the width of the line corresponding to state l. These quantities are thus connected, through $\delta_l^{(0)}$, with the boundary conditions of the unperturbed system.

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Determination and Properties of Anisotropy in Paramagnetic Resonance Absorption (*).

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Summary. — In order to measure the anisotropy in paramagnetic resonance absorption in single crystals, a method has been devised which does not require initial knowledge of the directions of the principal magnetic axes, and by which it is possible to determine the principal values together with the directions of the axes. It is shown that under certain assumptions the characteristic surface of g is an ellipsoid, and that thus, with a proper choice of the axes, the following equation holds: $g = g_x\alpha^2 + g_y\beta^2 + g_z\gamma^2$, where α, β, γ stand for the direction cosines of $\boldsymbol{H}$. The method has been applied for completing previous measurements, thus rendering their interpretation possible.

1. – Anisotropy of g–values.

It is well known that, in the case of salts of elements of the iron group, the ground paramagnetic level has a spin degeneracy. In fact, following the conventional method [1] of representing the interaction between the paramagnetic ion and the non-paramagnetic neighbours in terms of an electric field of proper symmetry and intensity, the orbital degeneracy which would exist in the free ion according with the Hund rule, is removed. Stronger interactions can lead to a total spin value which is less than that expected on the basis of the Hund

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(1) J. H. Van Vleck: *Electric and Magnetic Susceptibility* (Oxford, 1932).

rule [2]. Finally, exchange interactions between close paramagnetic ions, always yield, as ground paramagnetic level, a spin multiplet.

Let us call Δ_n the distance between the ground paramagnetic orbital level, labelled by 0, and the n-th higher level. Neglecting the interactions with the nuclear magnetic and quadrupole moments, the spin degeneracy of the ground paramagnetic level in a magnetic field H, is lifted by the Hamiltonian:

$$\mathcal{H} = \beta \boldsymbol{H}\cdot(\boldsymbol{L} + 2\boldsymbol{S}) + \lambda(\boldsymbol{L}\cdot\boldsymbol{S}) \,. \tag{1}$$

Let us assume an arbitrary orthogonal system of coordinates and let $\psi_{n,m} = \varphi_n \chi_m$ be the state function of the n-th orbital state and the m-th spin state.

When the principal term in the Hamiltonian (1) is the one which contains H, it is always possible, because of the common degree of spin degeneracy of the orbital levels, to choose the χ_m's so as to have

$$\chi_m^* \boldsymbol{S} \chi_m = m\boldsymbol{u} \,. \tag{2}$$

here $\boldsymbol{u}$ is the unit vector in the direction of the field H, the disection cosines of which we will call α', β', γ'. Evidently, calling S the resultant spin value of the state, m can assume $2S+1$ values: $-S, -S+1, \ldots, S-1, S$.

The matrix of the Hamiltonian (1) in the ground multiplet contains only the diagonal terms $2m\beta H$. This is a consequence of the chosen representation for the spins, and of the fact that the orbital magnetic moment of this multiplet is zero.

The conventional perturbation method carried to the second order, yields as the energy values of the ground multiplet:

$$E_m = 2m\beta H + \sum_{n,m'} \frac{|\langle n, m'|\beta \boldsymbol{H}\cdot\boldsymbol{L} + \lambda \boldsymbol{L}\cdot\boldsymbol{S}|0, m\rangle|^2}{\Delta_n} \,. \tag{3}$$

Remembering that $\langle m'|m\rangle = \delta_{m,m'}$ whatever the orbital level, it follows:

$$E_m = 2m\beta H + \lambda\beta H \sum_n \frac{\{(\langle n|\boldsymbol{u}\cdot\boldsymbol{L}|0\rangle)^*(\langle n|\boldsymbol{L}|0\rangle\cdot\langle m|\boldsymbol{S}|m\rangle)\}}{\Delta_n} + \text{conj. compl.} \tag{4}$$

$$+ \beta^2 H^2 \sum_n \frac{|\langle n|\boldsymbol{u}\cdot\boldsymbol{L}|0\rangle|^2}{\Delta_n} + \lambda^2 \sum_{n,m'} \frac{|\langle n|\boldsymbol{L}|0\rangle\cdot\langle m'|\boldsymbol{S}|m\rangle|^2}{\Delta_n} \,.$$

The second term of equation (4), taking into account eq. (2), becomes

$$2m\lambda\beta H \sum_n \frac{|\langle n|\boldsymbol{u}\cdot\boldsymbol{L}|0\rangle|^2}{\Delta_n} \,, \tag{5}$$

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[2] J. H. VAN VLECK: *Ann. Inst. Poincaré*,, **10**, 57 (1948).

and this term gives the departure of the g-value from 2 and its anisotropy.

The term in H^2 in eq. (4), being independent of m, contributes with equal amounts to the energies of all levels of the ground multiplet, and it is thus of no importance in the present case, although, it does contribute to the susceptibility of the crystal (VAN VLECK [1]) with a temperature independent term.

The term in λ^2 of eq. (4) may give rise to a zero-field splitting of the ground multiplet, within the limits allowed by Kramers' theorem [3]. Calling E_{0m} the contribution given by this term, and neglecting the terms in H^2, which are independent of m, and setting

$$g = 2\left(1 + \lambda \sum_n \frac{|\langle n|\boldsymbol{u}\cdot\boldsymbol{L}|0\rangle|^2}{\Delta_n}\right), \tag{6}$$

eq. (4) reduces to:

$$E_m = E_{0m} + mg\beta H\,. \tag{7}$$

Hence, the resonance field for the $(m-1) \to m$ transition, is given by the following equation

$$H_{(m-1)\to m} = \frac{h\nu - (E_{0m} - E_{0,m-1})}{g\beta}\,. \tag{8}$$

Eq. (6) can obviously be rewritten:

$$g = A\alpha'^2 + B\beta'^2 + C\gamma'^2 - 2A'\beta'\gamma' - 2B'\gamma'\alpha' - 2C'\alpha'\beta'\,, \tag{9}$$

which shows that the characteristic surface of the g-values is always a second order surface. Excepting the case that the resonance field diverges for some particular directions, this surface is certainly an ellipsoid. Taking as reference coordinates the principal axes of the ellipsoid, eq. (9) becomes:

$$g = g_x\alpha^2 + g_y\beta^2 + g_z\gamma^2\,, \tag{10}$$

where α, β, γ, stand for the new direction cosines of $\boldsymbol{H}$. If the system has an axis of symmetry, eq. (10) becomes:

$$g_\theta = g_\parallel \cos^2\theta + g_\perp \sin^2\theta\,. \tag{11}$$

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[3] H. A. KRAMERS: *Proc. Acad. Sci. Amst.*, **33**, 959 (1930).

In spite of the apparent disagreement with the usually followed formula [4-6]

$$g_\theta^2 = g_\parallel^2 \cos^2\theta + g_\perp^2 \sin^2\theta \tag{11'}$$

the latter reduces to eq. (11) if the squares of the differences of the terms containing λ/Δ are neglected.

In the case of axial symmetry, the spectrum is centred on the value [6]

$$H = \frac{h\nu}{g\beta}.$$

The terms E_{0m} are, in turn, each represented by a quadratic form of the direction cosines of the applied magnetic field.

On this footing, the experimental problem is therefore reduced to the determination of the coefficients of the quadratic forms which determine the g-values and the differences $E_{0,m} - E_{0,m-1}$.

2. – Experimental Method.

Within the limits in which the Hamiltonian (4) holds, and in the case ($S = \frac{1}{2}$) in which the spectrum consists of a single line, or more generally, when having the system an axial symmetry, the spectrum is centred on the value

$$H = \frac{h\nu}{g\beta};$$

we wish to prove the following. The complete determination of the g-tensor namely the determination of the three semi-axes of the ellipsoid (9) can be reduced, for example, to 6 measurements of the resonance field, in three planes, chosen with the unique condition that they have to be orthogonal to each other; i.e., to the measurements of the semi-axes of the ellipses which are the intersections of the characteristic g-surface with these planes.

In fact, being

$$Ax^2 + By^2 + Cz^2 - 2A'yz - 2B'zx - 2C'xy = 1 \tag{12}$$

the equation of the characteristic g-surface, the characteristic g-ellipse on the

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(4) M. H. L. Pryce: *Nature*, **164**, 116 (London, 1949).
(5) B. Bleaney: *Phil. Mag.*, **42**, 441 (1951).
(6) B. Bleaney and K. W. H. Stevens: *Rep. on Progr. in Phys.*, **16**, 107 (1953).

plane xy, for instance, is given by

$$Ax^2 + By^2 - 2C'xy = 1 . \tag{13}$$

If we measure in this plane the maximum and minimum g-values, which we call respectively g_z and g'_z, we have:

$$\begin{cases} g_z = \frac{1}{2}\{A + B + [(A - B)^2 + 4C'^2]^{\frac{1}{2}}\} \\ g'_z = \frac{1}{2}\{A + B - [(A - B)^2 + 4C'^2]^{\frac{1}{2}}\} . \end{cases} \tag{14}$$

Analogous equations hold for the g_x and g'_x, g_y and g'_y values. The coefficients A, B, C, A', B', C', are then given by the equations:

$$A = u - u_x , \qquad 4C'^2 = v_z^2 - (u_x - u_y)^2$$

and so on, where

$$u_x = g_x + g'_x , \quad v_x = g_x - g'_x , \quad \text{etc.}, \quad 2u = u_x + u_y + u_z . \tag{16}$$

The principal g-values are then the roots of the equation:

$$|\Delta - \varepsilon I| = 0 ,$$

where Δ is the matrix of the coefficients of the second order characteristic g-surface.

Furthermore, by the ordinary methods of analytic geometry, the directions of the axes relative to the three orthogonal planes on which the measurements have been carried out, may easily be determined.

In a future work, an attempt will be made to extend this method to more general cases, and to the investigation of the splittings.

3. – Application to the Case of $CuSO_4 \cdot 5H_2O$.

This method has been used for the determination of the g-values of a single crystal of $CuSO_4 \cdot 5H_2O$, at room temperature and at the liquid nitrogen temperature.

As has been shown by previous measurements on powders (7), the g-value

(7) M. B. Palma Vittorelli, M. U. Palma, D. Palumbo and M. Santangelo: *Ric. Sci.*, **25**, 1423 (1953).

of the powder changes from $g = 2.20$ to $g = 2.22$, in passing from room to liquid air temperature.

It is worth noticing that, while the width of the curve may strongly depend on the temperature through τ_l, the g-value is very little affected by this dependence (at least untill $\tau_l \nu \ll 1$) ([8]). It would seem therefore likely that a temperature dependence of the g-value could be mainly explained by the temperature dependence of the dimensions of the elementary cell. In the model of a mainly ionic interaction, these dimensions are involved with their inverse sixth and fourth powers in the expression of the electric field. According to this model, the g-value should in this case, decrease with the temperature, at least as far as no change in the symmetry of the electric field occurs. Such a change should however show similar effects on the g-values of the single crystal.

In order to clarify this point, analogous measurements on the single crystal have been carried out.

The elementary cell of $CuSO_4 \cdot 5H_2O$, as is well known ([9]), contains two paramagnetic ions, each of which may be thought subjected to an electric field of tetragonal symmetry, due to the water molecules and to the oxygen ions of the SO_4^{--} group which surround the paramagnetic ion. The two Cu^{++} ions are not equivalent, since the symmetry axes of the respective electric fields are 82° apart.

In spite of this fact, in every direction, the spectrum shows one single absorption line, up to fields of the order of 8000 gauss ([10]) and this happens because of the exchange interaction between the two paramagnetic ions in the elementary cell ([6,9,10]).

Nevertheless the g-value related to the resultant line is the arithmetic mean of the g-values that one would expect for each ion in the absence of exchange interaction, if account is taken of the different directions of the tetragonal axes of the crystalline electric field, with respect to the applied magnetic field (Van Vleck, quoted by Arnold and Kip ([9])). Thus, its characteristic surface is still, according with eq. (10), an ellipsoid. It is worth to remark that this does not happen in general if a formula of the type of eq. (11′) is used, instead of eq. (10).

For obvious symmetry reasons, the axes of the resultant ellipsoid will lie along the two bisectors of the angles formed by the two axes of the two ions, and along the perpendicular to their plane. The characteristic ellipsoids of the two individual g's are ellipsoids of revolution, while the one relative to the g-value of the cell does not retain this property.

([8]) J. H. Van Vleck and V. Weisskopf: *Rev. Mod. Phys.*, **17**, 227 (1945).
([9]) R. D. Arnold and A. F. Kip: *Phys. Rev.*, **75**, 1199 (1948).
([10]) D. M. S. Bagguley and J. H. E. Griffiths: *Proc. Roy. Soc.*, A **201**, 366 (1950).

4. – Experimental Results and Discussion.

The measurements have been carried out using the equipment of the Research Laboratory of Electronics of the Massachusetts Institute of Technology.

The equipment allowed the recording of the derivative of the resonance spectrum by means of a small amplitude modulation of the magnetic field ([11]). The magnetic field was electronically stabilized, and measured by means of a proton resonance signal; the klystron frequency was electronically locked to the resonance frequency of the cavity, and it was possible to measure it by comparison with a frequency standard.

The sample was glued in an arbitrary position on a small polystyrene cube, three orthogonal faces of which were successively placed in a position perpendicular to the rotation axis of the holder.

In each of the three planes thus examined, measurements along the directions of the maximum and minimum g-values were carried out. Along these six directions, the magnetic field values were measured corresponding to the zero—and maximum—slope of the absorption intensity.

The microwave frequency was about 9000 MHz.

The g-values and the maximum slope half width, in the six directions, at room temperature, are given in Table I, and the analogous values at liquid nitrogen temperature are given in Table II. By the above described method, the principal g-values have been calculated, and they are given in Table III.

Although the changes of the g-values with the temperature are of the order of the experimental error, the consistency of the results allows us to state that the uniform decrease of .003 is physically significant, at least as an order of magnitude.

These results prove that, according with the predictions, the mentioned displacement of the position of the maximum of the powder resonance curve must be ascribed to other reasons.

As shown in Table I and II, in passing from room to liquid nitrogen temperature, the line-width decreases anisotropically. The more one moves away from one of the directions of the principal axes, the larger is the decrease. The line intensity must therefore increase correspondingly. Thus the maximum of the powder absorption line will not be affected by the variation of the g-values of the single crystal as much as by the variations of the line-width and consequently of the intensity in the various directions.

Since, in the directions of maximum narrowing, that is of maximum inc-

([11]) R. Malvano and M. Panetti: *Nuovo Cimento*, **7**, 28 (1950).

TABLE I. – *Maximum and Minimum g-values, as measured in three orthogonal planes, and related maximum slope half-widths.* $T = 300$ °K.

$g_x = 2.244 \pm .002$	$\Delta H_x = 54$	Gauss
$g'_x = 2.090 \pm .001$	$\Delta H'_x = 15$	»
$g_y = 2.258 \pm .002$	$\Delta H_y = 56$	»
$g'_y = 2.092 \pm .002$	$\Delta H'_y = 15$	»
$g_z = 2.267 \pm .001$	$\Delta H_z = 30$	»
$g'_z = 2.228 \pm .002$	$\Delta H'_z = 44$	»

TABLE II. *Maximum and minimum g-values, as measured in three orthogonal planes, and related maximum slope half-widths.* $T = 77$ °K.

$g_x = 2.243 \pm .001$	$\Delta H_x = 35$	Gauss
$g'_x = 2.087 \pm .002$	$\Delta H'_x = 12$	»
$g_y = 2.254 \pm .001$	$\Delta H_y = 36$	»
$g'_y = 2.088 \pm .001$	$\Delta H'_y = 11$	»
$g_z = 2.264 \pm .001$	$\Delta H_z = 22$	»
$g'_z = 2.226 \pm .002$	$\Delta H'_z = 28$	»

TABLE III. – *Principal g-values, calculated by the described method, using the data of Tables I and II.*

$T = 300$ °K	$g_1 = 2.267$	$g_2 = 2.236$	$g_3 = 2.086$
$T = 77$ °K	$g_1 = 2.264$	$g_2 = 2.233$	$g_3 = 2.083$

rease in the intensity, the g-values are 2.243, 2.254, and 2.226, the g of the powder shall be displaced in such a way to become closer to these values, which is in agreement with the experimental results previously reported (7).

* * *

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RIASSUNTO (*)

Per misurare l'anisotropia nell'assorbimento per risonanza paramagnetica nei monocristalli β stato ideato un metodo che non richiede la conoscenza preventiva delle direzioni degli assi magnetici principali e per mezzo del quale è possibile determinare i valori principali con le direzioni degli assi. Si dimostra che, ammesse determinate ipotesi, la superficie caratteristica di g è un ellissoide e che così, con un'appropriata scelta degli assi, valgono le seguenti equazioni: $g=g_x\alpha^2+g_y\beta^2+g_z\gamma^2$, dove α, β, γ sono i coseni direttori di $\boldsymbol{H}$. Il metodo è stato applicato per il completamento di precedenti misure, rendendone così possibile l'interpretazione.

(*) *Traduzione a cura della Redazione.*

Gap Density Measurements in Nuclear Emulsions.

F. T. Gardner and R. D. Hill

Physics Department, University of Illinois (*) - *Urbana, Ill.*

(ricevuto il 2 Agosto 1955)

Summary. — The technique of gap counting has been extended by using measurements of the variation of the number of gaps with range. This extension is particularly useful in heavily developed G5 emulsions.

In machine exposures where background radiations are often of high intensity, it is sometimes expedient to use heavily developed emulsions in order to render light particles outgoing from K-particles and hyperons more apparent in scanning. Although heavy development has made grain counting of ending particles more difficult at short ranges, this has not generally been of importance since the particles could be provided with adequately long ranges by directing them along the lengths of the emulsion strip (1). In some cases, however, tracks which have emerged from stars produced in the emulsion were too short for grain density measurements and it was then necessary to resort to gap length measurements. The present note is concerned with some features that have become apparent in the method of gap counting in heavily developed emulsions.

Proceeding in the usual way (2) of determining gap lengths, it was observed that on account of the heavy development of the emulsions, the numbers and lengths of gaps starting from the ends of the tracks were so small and the

(*) Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission. Assisted also by a grant from the University of Illinois, Graduate College.

(1) R. D. Hill, E. O. Salant and M. Widgoff: *Phys. Rev.*, **99**, 229, (1955)

(2) M. G. K. Menon and C. O'Ceallaigh: *Proc. Roy. Soc. Lond.*, A **221**, 292 (1954).

consequent statistical fluctuation so large that no clear separation of K-particles and protons was possible for less than 2-3 mm of track length. Examples of such observations are given in Fig. 1 where gap lengths per 100 μm of track are plotted vs range. Included on the same plot are three π-mesons (solid points), two K-mesons (open circle points) and two protons (solid triangle points). A minimum gap-length cut-off of 30 divisions (0.64 μm) was used, and for our emulsions the mean gap length for plateau-ionization tracks was 3.0 μm. A best line was fitted to the pion points and the K-meson and proton lines were drawn according to the known mass ratios of these particles to the π-meson.

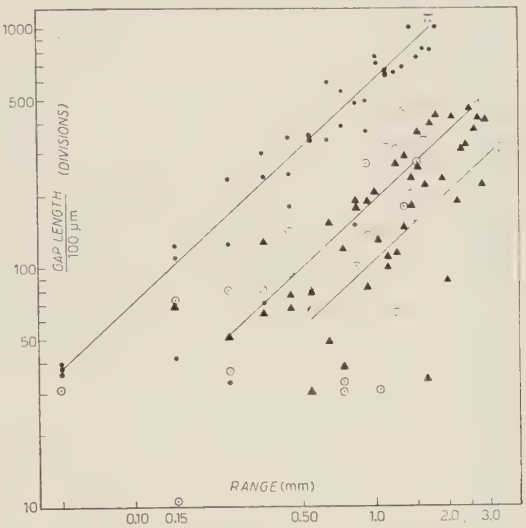

Fig. 1.

Instead of continuing gap length measurements to large ranges, numbers, only, of gaps greater than a certain cut-off minimum were measured. This is essentially the same as counting blobs, except that by imposing a condition of a minimum gap length on the counting, all the advantages of compensating for finite resolving power losses and for subjective observational errors, as well as of allowing for dipping tracks, are gained. Examples of such observations taken in this way are given in Fig. 2, where three cases of π-mesons, three cases of K-particles and two cases of protons are plotted together. Each point represents the number of gaps (per 100 μm) greater than 30 divisions (0.67 μm) in length. It is clear from Fig. 2 that, although the statistical errors are still large (*), there is good discrimination between the different particles, and expecially when counted at those ranges for which the gap densities are approximately the same. The ease of counting merely the number of gaps is a great advantage.

Curves relating blob and grain densities vs range in G5 emulsions, for different degrees of development, have been given by O'CEALLAIGH [3]. The proton curve of Fig. 2 and the grain count curve for protons [4] agree with

(*) The variation of the number of gaps per 100 μm from the mean is everywhere only of the order or less than the square root of the mean number of gaps.

[3] C. O'CEALLAIGH: *Suppl. Nuovo Cimento*, **12**, 412 (1954).

[4] R. D. HILL, F. T. GARDNER and J. E. CREW: *Nuovo Cimento* **2**, 824 (1955).

O'Ceallaigh's curves for heavily developed emulsions and an observed minimum grain count of approximately 36 grains per100 μm. For a heavily developed

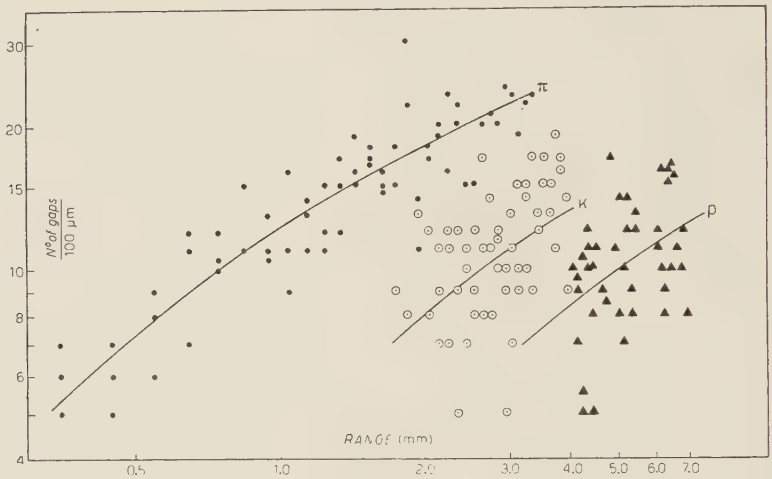

Fig. 2.

emulsion of this minimum grain density, gap or blob densities vs range are the most suitable measure of ionization densities of protons in the range inter-

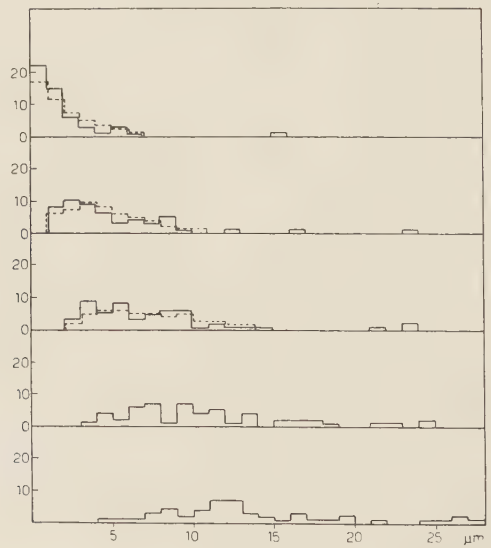

Fig. 3.

val from 1 to 10 mm, whereas above 1 cm range, grain density measurements are more suitable.

Incidental to the present study of gap lengths, we have made a somewhat novel test of the validity of the law ([2]) of distribution of gap widths. In a random sample of approximately 50 μ-e decays, we have measured gaps between the ends of μ-meson tracks and the 1st, 2nd ... etc., grains of their outgoing electron tracks. These results are shown in Fig. 3, where numbers of cases of gap widths in one micron intervals are plotted vs gap widths for the first five grains of all tracks. The observed histograms are, within statistics, precisely what one should expect from a perfectly random process of producing grains along a track. On the basis of a probability distribution: $P(x) = (1/\bar{G}) \cdot \exp[-(x/\bar{G})]$ where x is the length of a gap and $\bar{G}$ is the mean gap length (for our emulsions and an electron track $\bar{G} = 3.0$ μm), the expected distribution of separation between μ-meson tracks and the 1st, 2nd and 3rd grains are shown by the dashed-line histograms in Fig. 3.

RIASSUNTO (*)

La tecnica del conteggio delle lacune è stata estesa ricorrendo a misure della variazione del numero di lacune col range. Questa estensione si dimostra particolarmente utile nelle emulsioni G5 fortemente sviluppate.

(*) *Traduzione a cura della Redazione.*

Evidence for Nuclear Interaction of a Charged Hyperon in Flight.

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(ricevuto il 2 Agosto 1955)

Summary. — A Σ^- hyperon, arising from a 12 + 0p star, disappears in flight, presumably by interacting with a proton and producing a neutral hyperon and neutron.

An event, interpreted as the disappearance of a charged Σ-hyperon in flight, has been observed in a stack of 400 μm G5 emulsions exposed in the Brookhaven cosmotron. The hyperon track originates in a twelve prong proton star and ceases abruptly at a distance of 23.5 mm from the star and after passing through four emulsion strips. At the point where the track suddenly ceases it is clear from gap density and scattering observations that the hyperon is not at the end of its range. The identity of the hyperon has been established by the following measurements.

1. – Gap density-range.

Meson and proton tracks in emulsions can be clearly distinguished from one another by counting the number of gaps vs range (1). The number of gaps per 100 μm of the hyperon track were measured from the end where the track ceased. These points for 0.5 mm intervals are plotted as open circles in Fig. 1. The expected line for the Σ-hyperon variation of gap count vs range in our

(*) Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission. Assisted also by a grant from the Graduate College of the University of Illinois.

(1) F. T. Gardner and R. D. Hill: *Nuovo Cimento* **2**, 820 (1955).

emulsion is also shown in Fig. 1. It is clear that the abscissa values of the measured points should be increased by an additional range in order to bring about agreement in both position and slope of measured points with the Σ-hyperon line. The full circle points of Fig. 1 are for an additional 3.5 mm applied to the range as measured from where the track ceased. On the basis of gap density measurements alone it is not possible to decide whether the track is a hyperon or proton.

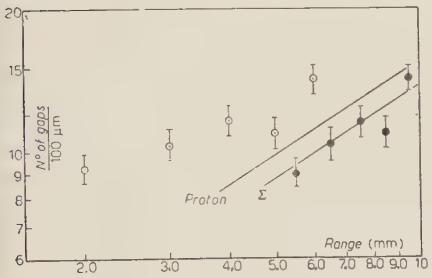

Fig. 1.

2. – Grain density-range.

The grain density of the hyperon track was determined near its origin in the star, i.e. 20 to 23 mm from the end where the track ceased. A plot of the number of grains per 100 µm, averaged over sections of 0.5 mm of track, was then made against the observed range, plus 3.5 mm extra range which was estimated from gap density measurements. Calibration of the grain density vs. range variation was made using protons, and in particular one proton which crossed over the hyperon track in a region of comparable grain density. These results are shown in Fig. 2. The mass of the hyperon determined in the same manner as used previously [2] for K-particles is $(2360 \pm 120)\ m_e$. It should be pointed out that a proton of the observed grain count of the hyperon track at its origin would already end at 21 mm.

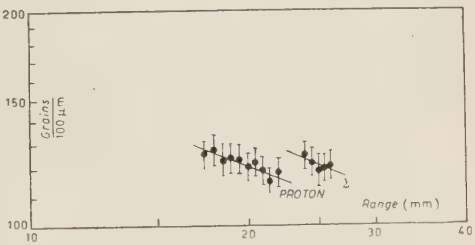

Fig. 2.

3. – Scattering-grain density.

The grain density of the hyperon track near its origin is 3.0 ± 0.2 times the plateau grain density of ~40 grains per 100 µm. In our emulsions this corresponds to a Σ-particle of approximately 100 MeV ($p\beta = 190$ MeV/c) or alternatively to a proton of approximately 80 MeV ($p\beta = 150$ MeV/c). Scattering of the same region of the track, using the standard 100 µm—200 µm cells method, yielded $\hat{\alpha}_{100\mu m} = 0.131^\circ \pm 0.022^\circ$. For a scattering constant [2]

(2) R. D. Hill, E. O. Salant and M. Widgoff: *Phys. Rev.*, **99**, 299 1955.

of 25.8, the observed $p\beta$ of the track is therefore (190^{+46}_{-21}) MeV/c, in very good agreement with the expected value for a Σ-hyperon. The expected range [3] of a 100 MeV Σ-hyperon is 26 mm, whereas that of an 80 MeV proton is only 21 mm. An unobserved additional range of 2.5 mm for a Σ-hyperon is clearly consistent with the gap density measurements.

4. – Scattering-range.

The hyperon track was scattered in each emulsion strip and the observed average values of $\hat{\alpha}_{100\,\mu m}$ are plotted in Fig. 3 where the values of the ranges are the observed averages in each strip plus the extra 3.5 mm that the hyperon could have gone to the end of its range. The expected variations of $\hat{\alpha}_{100\,\mu m}$ for Σ-hyperon and proton are shown by the lines in Fig. 3. Although the agreement with the hyperon line is not strikingly good it is nevertheless much more consistent with the event being associated with a hyperon than a proton. An additional 5 to 8 mm of track would be required to bring about agreement of the scattering results with the proton line, and if this were so the total range of the proton at the star would be approximately 30 mm. A proton of this range would have a grain density of 100 grains per 100 μm in our emulsions, a value which is 20% lower than the observed grain density of 125 grains per 100 μm. Since the observed statistical deviation of the grain count is only ~ 2 to 3%, such a low value is inadmissible.

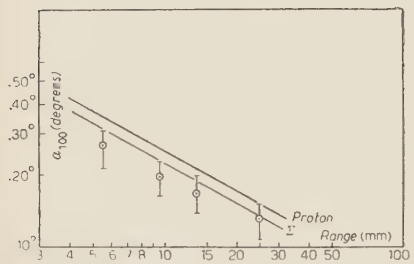

Fig. 3.

4. – Interaction Process.

This event is probably to be identified as either a nuclear or charge-exchange interaction of a negative Σ-hyperon in flight. Either of the following interactions would be consistent with the ideas of associated production and strong interactions [4,5].

$$\Sigma^- + N^+ \rightarrow \Lambda^0 + N^0 \quad \text{(nuclear interaction)}$$

$$\Sigma^- + N^+ \rightarrow \Sigma^0 + N^0 \quad \text{(charge exchange)}.$$

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[3] H. Fay, K. Gottstein and K. Hain: *Suppl. Nuovo Cimento*, **11**, 234 (1954).
[4] M. Gell-Mann and A. Pais: *Glasgow Conference Paper* (unpublished).
[5] R. G. Sachs: *A classification of fundamental particles* (unpublished).

As the hyperon had approximately 30 MeV at the instant of its disappearance, it seems unlikely that it could have been captured by a nucleus via a Bohr orbit and have given rise to a zero prong star. The time of flight ([3]) of the hyperon in its own reference frame was $2.0 \cdot 10^{-10}$ s. The available time of flight ([6]) was infinite. We have been unsuccessful in observing any evidence of either Σ^0, γ or Λ^0 which might have resulted from the hyperon reaction.

It is interesting to note that the present event represents the first nuclear interaction of a charged hyperon in 23.25 cm of path (*). At the time of the Padua report ([7]), a total path length of 20.9 cm, representing ranges either to the point of decay in flight or to decay or absorption at rest, had been observed in 33 cases of charged hyperons. Although it is not very meaningful to specify a cross-section on the basis of one event, it is tempting to consider the evidence as suggesting that the nuclear interaction cross-section for charged hyperons is not very different from that for charged pions in emulsion.

([6]) C. Castagnoli, G. Cortini and C. Franzinetti: *Suppl. Nuovo Cimento*, **12**, 297 (1954).

(*) We confine our attention to ending hyperons for which the main body of data has been accumulated. Evidence for the nuclear absorption of a high energy hyperon of approximately 300 MeV, with the production of an associated K-particle, has been given recently by Block and King: *Phys. Rev.*, **97**, 1415 (1955).

([7]) *Report of the Commitee on Charged Hyperons, Suppl. Nuovo Cimento*, **12**, 448 (1954).

RIASSUNTO (*)

Un iperone Σ^- originato da una stella 12 + 0p scompare in volo, presumibilmente per interazione con un protone con produzione di un iperone neutro e un neutrone

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(*) *Traduzione a cura della Redazione.*

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A Probable Example of the Reaction:

$$\pi^- + p = K^+ + K^- + n.$$

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(ricevuto il 10 Settembre 1955)

Summary. — An event is described in which a fast singly charged particle, produces a star from which the only charged particles emitted are a negative K-meson and another particle of ~ 1000 electronic masses. The interest of the event is due to the fact that this is the first case in which a mechanism of creation of a negative K-meson can be specified with a good degree of reliability. The event is shown to be in agreement with the selection rules resulting from associated production ideas.

1. – Introduction.

During the examination of the parent stars of negative K-mesons found in the G-stack [1], a nuclear interaction was found from which emerged a negative K-meson accompanied by only one other charged particle. The latter escaped from the stack without coming to rest but from combined measurements of ionization and multiple scattering on the track it appears to have been due to a charged K-meson.

The interaction was produced by a lightly ionizing particle originating in a large nuclear disintegration.

The event appears worthy of particular discussion in so far as it is the second example of the association of another unstable particle in the pro-

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(1) *Observations on heavy meson secondaries.* Universities of Bristol, Dublin Adv. Stud., Dublin Univ. Coll., Genova-Milano, Padova: *Reports of the Pisa Conference,* 1955.

Plate 1.

duction of a negative K-meson hitherto reported [2], and the first case in which the number of particles involved is small enough to allow the details of the interaction to be fully specified with a good probability.

2. – Experimental Details.

A microphotograph of the complete event is shown in Plate 1. From a star, X, of type 10 + 16p, emerged a lightly ionizing particle, A, which traversed a length of 108 mm before causing the interaction indicated at Y, in which were produced the two particles B and C. The particle producing track B, came to rest after a range of 6.4 mm in three plates and gave rise to a typical negative K-meson capture star of four prongs, three black and one thin. The other track, C, was traced to the edge of the stack and the length observed was 16.5 mm in two plates. No other tracks could be found to originate in the interaction at Y.

The results of the measurements made on the three tracks A, B, C are shown in Table I.

TABLE I.

	Track A	Track B	Track C
Length (mm)	108 (obs.)	6.39 (tot.)	16.5 (obs.)
Length per plate (mm)	5.4	2.5	8.0
Azimutal angle respect to A		4° 35′	21° 45′
Angle with the plane of the emulsion	3° 20′	9° 50′	— 3° 25′
I/I_0	1.01 ± .02		1.28 ± .03
$p\beta$ (MeV/c)	1650^{+500}_{-300}		430 ± 50
Supposed identity	π^-	K^-	K^+
Energy at emission (MeV)	1 500	31	210

Some of these measurements require special discussion since the interaction denoted by Y, occurred only 16 mm from the side edge of the stack. In the

(2) M. W. FRIEDLANDER, D. KEEFE and M. G. K. MENON: *Nuovo Cimento*, **2**, 466 (1955).

neighbourhood of the edges of the emulsion sheets composing the stack, generally one expects to find a decrease in the developed grain density and an increase in the measured multiple scattering, compared to central regions of the plate. Whether mass measurements are made by the ionization v. scattering, scattering v. range, or ionization v. range methods, these effects will result in a spurious underestimation of the mass of the ionizing particle. Therefore we discuss firstly in some detail the measurements made in establishing that tracks B and C were indeed caused by K-mesons and were neither due to heavier particles. Later are treated the details of the measurements leading to an estimate of the nature and identity of the primary particle of the event Y.

(i) *Measurements on Track* C. – In order to evaluate the effects due to distortion near the edge of the emulsion on the multiple scattering, six tracks were chosen for measurement which crossed the two emulsion strips containing track C. All these tracks had a dip angle similar in magnitude and sign and were close and roughly parallel to track C, and were found to be produced by proons with energies between 300 and 500 MeV. Measurements of multiple scattering were made over a length of several centimetres using these tracks, in each case beginning at a distance of 1.5 mm from the processed edge of the emulsion. Next imaginary « cuts » were made at distances 1.5 mm, 2.5 mm, 3.5 mm etc. from and parallel to the edge and the average over the six tracks evaluated for the second differences $\bar{\Delta}''$. In Fig. 1 are shown the values of $\bar{\Delta}''_{300\mu m}$ (corrected for noise) computed in each of these 1 mm wide strips, plotted as a function of the distance from the edge of the corresponding strip. One can see that the scattering measurements on track C are expected to give a reliable estimate of the true multiple coulomb scattering in the track provided that measurements are confined to a region further than about 4 mm from the edge of the plate. Attempts to obtain further information from the application of the third (and higher) difference method to the scattering measurements obtained in the 4 mm border, were unsuccessful, indicating a rather complex form of distortion.

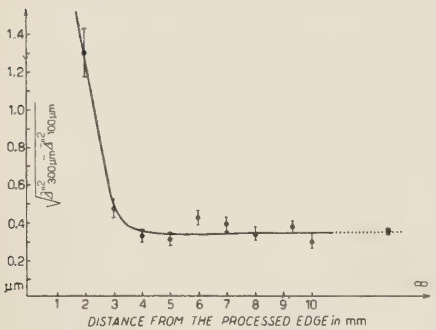

Fig. 1. – Variation of the apparent scattering of tracks with the distance from the processed edge of the emulsion.

The variation in the plateau ionization as one proceeds towards the edge of the emulsion has been determined by measurements of grain density in the tracks of electrons arising in the decay of μ-mesons. It was found that

the plateau grain density had an almost constant value except nearer than about 5 mm to the edge, where it began to decrease appreciably.

Measurements of scattering and blob density, besides being made on track C, were also carried out on 11 calibration tracks all due to protons, and chosen to lie in a rectangle 3·4 cm² centred at Y and having one of its long sides coincident with the edge of the emulsion. All the calibration tracks had approximately the same geometrical disposition as had track C; namely, they were similar as regards dip, distance from and angle to the processed edge. In every track, 1 000 blobs were counted in each of the two plates containing track C always in a chosen layer of emulsion, and scattering measurements on each were made over a length of about 15 mm containing on the average 50 independent cells of 300 μm, per track. The scattering measurements quoted for track C referred to a segment 11.8 mm long containing 45 independent cells of 250 μm each.

As an additional check five further calibration tracks due to protons were selected for measurement, all under less favourable conditions than track C (i.e. they layd nearer the edge at an average distance of about 6 mm, were more steeply inclined to the emulsion plane, etc.). On each, the multiple scattering was measured over a length of 10 mm, and 1 000 blobs were counted in that segment of track lying nearest the edge.

In the evaluation of $\bar{\alpha}_{100\mu m}$ for all tracks from the values obtained, noise has been eliminated in the usual way, as has also been the contribution from « spurious scattering » (c.f. the measurements on track A). The effect of this latter correction has been to diminish the value of $\bar{\alpha}_{100\mu m}$ for each track by between 5% and 10%.

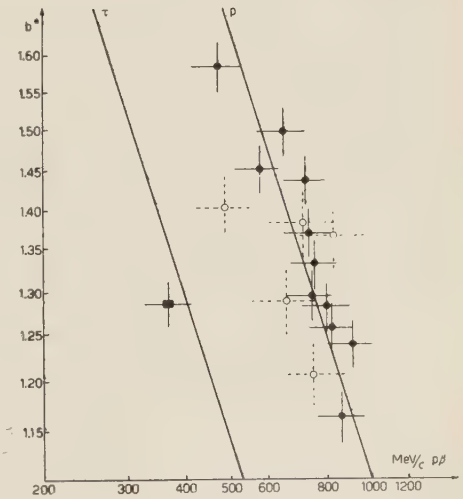

Fig. 2. – $p\beta$ versus normalized blob density plot for track C and for 16 calibration tracks: ■ track C; ● calibration protons of the 1-st group; ○ calibration protons of the 2-nd group.

Fig. 2 shows the values of the normalised blob density and $p\beta$ for track C (black square), for the 11 calibration tracks of the first group (black circles), and the 5 calibration tracks of the second group (open circles). The line marked « p » is the « least squares » best fit to the proton calibration points. The mass spectrum derived from this (b^*, $p\beta$) diagram is shown in Fig. 3.

From what has been said it appears that in the absence of a very severe statistical fluctuation, the particle producing track C could not have had

protonic mass and was in fact a K-particle. Had any systematic errors affecting either the ionization or multiple scattering measurements on track C been present it is very unlikely that they would not have also affected at least some of the calibration measurements in a similar way.

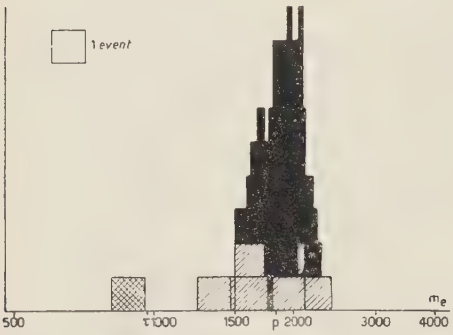

Fig. 3. – Mean spectrum of the particles of Fig. 2: ▩ track C; ■ calibration protons of the 1-st group; ▨ calibration protons of the 2-nd group.

(ii) *Measurements on Track* B. – The estimation of the mass of the particle producing track B is a great deal simpler since it came to rest within the emulsion. In view of what has been said previously concerning the investigation of distortion near the edge of the plate, measurements of constant-sagitta multiple scattering versus range are expected to be quite reliable; in fact they yield a mass estimate of $760^{+200}_{-150}\ m_e$.

In the mass determination by the ionization versus range method, a comparison was made with 14 other tracks produced by particles of known mass, and each was measured at a range such that its ionization was close to that in track B. In all tracks the grain density was estimated by counting 2000 grains at the same depth in the emulsion and at the same distance from the edge. The mass spectrum obtained in this way is shown in Fig. 4. The best value for the mass of the particle producing track B found by this means is $940 \pm 100\ m_e$.

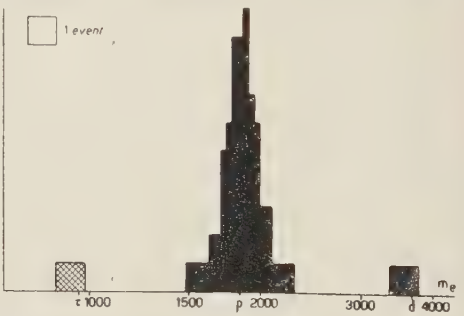

Fig. 4. – Mass spectrum of particle B, of 13 calibration protons and of one deuteron obtained from ionization versus range measurements: ▩ track B; ■ calibration tracks.

Further support for the identification of this particle as a negative K-meson is provided by the nature of the capture star produced at the end of its range. This is formed by three heavy and one thin track, the latter being most probably caused by a π-meson, and the total amount of energy produced in visible form is estimated to be about 210 MeV, a typical value arising in K^--capture [3],

[3] M. W. Friedlander, Y. Fujimoto, D. Keefe and M. G. K. Menon: *Nuovo Cimento*, **1**, 90 (1955).

but rather too high for an Y^--capture ([4]).

TABLE II.

	Prongs of the K^--capture star			
	1	2	3	4
Identity	(π)	p	p or α	p or α
Range (mm)	> 15	.44	.040	.013
Energy (MeV)	~ 60	9	~ 2	~ 1

(iii) *Measurements on Track* A. – The determination of the energy of the primary particle presented a more difficult problem because of the higher momentum involved and the presence of appreciable « spurious scattering » ([5]) in the plates which it traversed. We have noticed that the effect of this type of distortion, which cannot be eliminated in the usual way used for « noise », is to yield an answer of about 1.5 GeV/c for the apparent $p\beta$ of a particle of any true $p\beta$ greater than this value. Since it is of interest to establish an upper limit to the energy of the primary particle of event Y, we have employed a correction procedure which, although capable of further refinement, appears to be quite reliable.

In 15 of the 20 plates containing track A, tracks were selected of corresponding geometry i.e. not more than 1 cm away from track A and within an angle of 30° to it with the same dip angle; each of these were due to particles of certainly greater than 10 GeV since all produced high energy showers within the stack. The results of scattering measurements on track A and on four such calibration tracks, averaged in two groups of plates, are shown in Table III.

It appears that the multiple scattering in Track A is appreciably greater than that in the high energy calibration tracks and that one can obtain a significant answer by eliminating the spurious scattering provided it is assumed to play the part of an independent variable. Since the spurious scattering probably does not obey a normal distribution ([6]) one is not justified in using

([4]) M. CECCARELLI, N. DALLAPORTA, M. GRILLI, M. LADU, M. MERLIN, G. SALANDIN and B. SECHI: *Reports of the Pisa Conference*, 1955.

([5]) S. BISWAS, B. PETERS, F. A. Sc. and RAMA: private communication. Submitted to: *The Proceedings of the Indian Academy of Sciences*.

([6]) S. BISWAS F. C. GEORGE, B. PETERS and M. S. SWAMI: *Suppl. Nuovo Cimento*, **12**, 369 (1954).

TABLE III (*).

	Plates 31-38		Plates 39-45	
	Tracks of energy > 10 GeV	Track A	Tracks of energy > 10 GeV	Track A
$\overline{\Delta'''}_{200\ \mu m}$ (**)	$.15 \pm .01$	$.20 \pm .02$	$.18 \pm .01$	$.27 \pm .02$
$\overline{\Delta'''}_{400\ \mu m}$	$.33 \pm .04$	$.43 \pm .05$	$.32 \pm .03$	$.54 \pm .06$
$\sqrt{\overline{\Delta'''^2}}_{400\ \mu m}$	$.42 \pm .03$	$.50 \pm .04$	$.40 \pm .04$	$.59 \pm .06$

(*) Sagittae are given in μm.

(**) We confine our discussion to the 3-rd differences because they are those for which the ratio signal to noise is most significant.

the usual elimination formula:

$$\overline{\Delta'''^2}_{true} = \overline{\Delta'''^2}_{measured} - \overline{\Delta'''^2}_{spurious}.$$

We believe however, that instead one can use an analogous formula in which the arithmetic mean values are replaced by the root mean square values, viz.

$$(1) \qquad \overline{\Delta'''}_{true} = \frac{1}{1.06}\sqrt{\frac{2}{3}}\sqrt{\frac{2}{\pi}}\sqrt{\overline{\Delta'''^2}_{measured} - \overline{\Delta'''^2}_{calibration\ tracks}}$$

in which it is assumed only that the Δ'''_{true} obey a normal distribution. It is clear that this procedure also results in the elimination of ordinary noise.

In order to evaluate the sensitivity of this subtraction procedure we have calculated the values of $p\beta$ given by formula (1) for different assumed values of $(\overline{\Delta'''^2}_{meas})^{\frac{1}{2}}$ and using for $(\overline{\Delta'''^2}_{cal.\ tr.})^{\frac{1}{2}}$ either of the limit values obtained from the calibration measurements. These values lie within the area marked by horizontal shading in Fig. 5. The two straight lines parallel to the x-axis enclosing the vertically shaded area, represent on the other hand the limits of $(\overline{\Delta'''^2}_{meas})^{\frac{1}{2}}$ found for track A; the domain of intersection of these two zones therefore indicates the acceptable values for the $p\beta$ of the primary particle obtained from this elimination procedure. The two broken lines in the same figure show the values of $p\beta$ calculated from $(\overline{\Delta'''^2})^{\frac{1}{2}}$ without noise elimination

(dashed line) and with noise elimination using the usual method (dotted and dashed line).

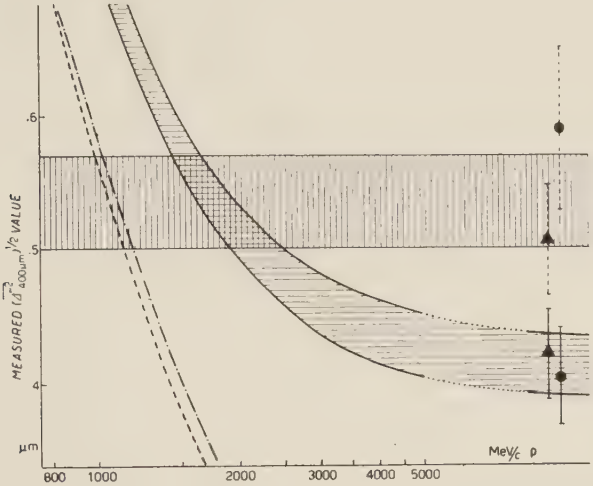

Fig. 5. – Apparent sagitta values resulting for tracks of various $p\beta$, as an effect of « spurious scattering »: $\overline{\mathrm{I}}$ track A; $\underline{\mathrm{I}}$ very energetic calibration tracks; ▲ 1-st group of plates (31-38); ● 2-nd group of plates (39-45).

The value of $p\beta$ for track A is then 1650^{+500}_{-300} MeV/c where the errors have been calculated from the formula given by GOLDSCHMIDT (7) and are in agreement with those deduced from the width of the common domain (cross hatched) in Fig. 5.

The grain density in track A was measured by counting about 5000 grains in several plates and was compared with that in tracks known to be due to particles at plateau ionization. Fig. 6 shows the result of the $(g^*, p\beta)$ measurements on track A compared with the

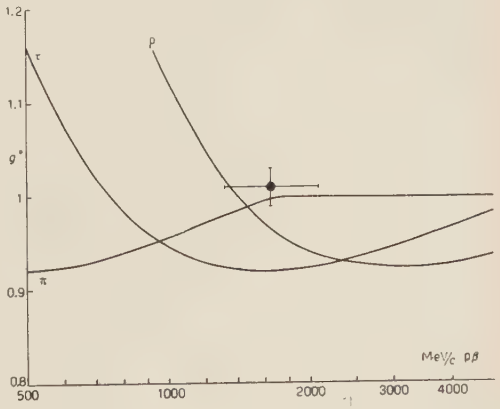

Fig. 6. – $p\beta$ versus normalized grain density for particles at nearly minimum ionization [E. PICKUP *et al.*,(8)]. The point corresponds to track A.

(7) Y. GOLDSCHMIDT-CLERMONT: *Nuovo Cimento*, **7** 331 (1950).

(8) E. PICKUP and L. VOYVODIC: *Phys. Rev.*, **85**, 91 (1952); M. DANYSZ, W. O. LOCK and G. YEKUTIELI: *Nature*, (London), **169**, 364 (1952); R. R. DANIEL, J. H. DAVIES, J. H. MULVEY and D. H. PERKINS: *Phil. Mag.*, **14** 753 (1952); M. BALDO, G. BELLIBONI, B. SECHI and G. T. ZORN: *Suppl. Nuovo Cimento*, 12, 220 (1954), *Reports of Padua Conference*.

expected curves due to Voyvodic and Pickup for different masses. Assuming in the present case the validity of these curves which have been checked by several Authors ([8]) we conclude that track A was produced by either a π-meson or a proton but was not due to a K-meson. At the measured grain density a K-meson would have had a $p\beta$ either less than about 1 GeV/c or greater than 4 GeV/c. A K-particle with a $p\beta$ greater than 4 GeV/c would show multiple scattering indistinguishable from that in the tracks of the calibration protons (see Fig. 5) whereas that in track A was clearly significantly greater. That track A was caused by a proton appears rather unlikely both by charge conservation and by a discussion of the dynamics of the interaction at Y; it seems most likely that in fact it was due to a π-meson.

3. – Discussion.

Because of the obvious lack of charge conservation in the event indicated at Y, it must be concluded that it was due to a nuclear interaction and not a spontaneous decay.

Accepting the proposed identification of the particles producing tracks A, B and C, the nuclear interaction giving rise to the observed event can be expressed in the following general way:

$$\text{(2)} \qquad \mathrm{A} + \mathrm{V} = \mathrm{K}^+ + \mathrm{K}^- + \mathrm{W}\,,$$

where V indicates the target nucleon or nucleons and W includes a number of nucleons and hyperons — governed by the law of conservation of baryons — and also a number of neutral π- and K-mesons. Neglecting for the moment any theoretical models which might suggest or forbid certain reactions, we can consider the following special cases of (2):

$$(a) \qquad \mathrm{p} + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + (\mathrm{n}+1)\mathrm{N}\,,$$

$$(a') \qquad \mathrm{p} + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + \mathrm{Y} + \mathrm{nN}\,,$$

$$(a'') \qquad \mathrm{p} + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + (\mathrm{n}+1)\mathrm{N}\ (\text{or } \mathrm{nN} + \mathrm{Y}) + 0^0\,,$$

$$(a''') \qquad \mathrm{p} + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + (\mathrm{n}+1)\mathrm{N}\ (\text{or } \mathrm{nN} + \mathrm{Y}) + \pi^0\,,$$

$$(b) \qquad \pi + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + \mathrm{nN}\,,$$

$$(b') \qquad \pi + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + (\mathrm{n}-1)\mathrm{N} + \mathrm{Y}\,,$$

$$(b'') \qquad \pi + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + \mathrm{nN}\ [\text{or } (\mathrm{n}-1)\mathrm{N} + \mathrm{Y}] + 0^0\,,$$

$$(b''') \qquad \pi + \mathrm{nN} = \mathrm{K}^+ + \mathrm{K}^- + \mathrm{nN}\ [\text{or } (\mathrm{n}-1)\mathrm{N} + \mathrm{Y}] + \pi^0\,.$$

We need not consider reactions in which a larger number of unstable particles appears on the right hand side of these equations since the primary particle had insufficient energy to produce them. Taking into account the possible distributions in charge it is clear that only reactions of type (b), (b'), (b''), and (b'''), viz.

$$(4)\qquad \begin{cases} \pi^- + p = K^+ + K^- + n\,, \\ \pi^- + p = K^+ + K^- + Y^0\,, \\ \pi^- + p = K^+ + K^- + n \text{ (or } Y^0) + \theta^0\,, \\ \pi^- + p = K^+ + K^- + n \text{ (or } Y^0) + \pi^0\,, \end{cases}$$

can give a number of ionizing prongs less than or equal to the observed number: these could therefore have taken place either with a hydrogen nucleus or with a heavier nucleus from which no other charged secondary products emerged.

However, reactions of the general types (a), (a'), (a''), (a''') and (b), (b'), (b''), (b'''),

$$(5)\qquad \begin{cases} a)\quad p + N = K^+ + K^- + p + N \text{ etc.}\,, \\ b)\quad \pi^- + N = K^- + K^- + p \text{ etc.}\,, \end{cases}$$

and a fortiori those giving a larger number of charged products could only have taken place in a nucleus heavier than hydrogen and must have been followed by the charge exchange of the produced proton or protons in order to agree with the observations. Though not a sufficient cause for excluding them, these requirements make such reactions appear much less likely to have been responsible for the observed event.

Considering the group of particles, W, to be a single body, we can now make an attempt to discriminate among all these possibilities, by calculating on dynamical grounds the mass and energy of the particle W allowed by the conservation laws, for various assumed values of the mass and energy of particle A. Fig. 7 shows four curves relating to reactions of the types (a), (b), (b') and (b''). At any value of $p\beta$

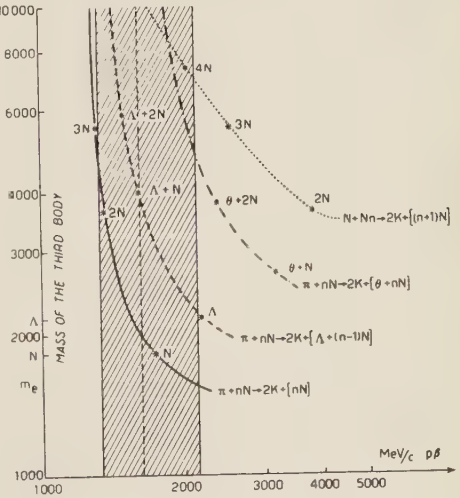

Fig. 7. – Diagram for calculating the mass of particle W from the $p\beta$ of particle A in a reaction of the type $A + V \to K^+ + K^- + W$ (see text).

shown on the abscissa, the corresponding ordinate indicates the mass value obtained for W which satisfied theconservation laws. The asterisks refer to values of $p\beta$ for which the mass of W corresponds to an integral number of nucleons, or to a hyperon plus an integral number of nucleons, that is, to certain allowed physical solutions. The dashed vertical line indicates the measured value of $p\beta$ for particle A, and the shaded area shows the limiting errors on this values.

If the primary particle were a nucleon (v. dotted curve) then the secondary « particle » W would have had a mass almost equal to that of four nucleons. The emission of such a large number of rather energetic nucleons (about 200 MeV apiece) would appear to be unlikely in view of the absence of charged particles other than the K-meson pair originating in Y. The same argument applies to reactions of type (a') where a Λ^0-particle takes the place of a neutron in the fragment W, and a fortiori for reactions of the type (a''). We therefore conclude that the reaction schemes (a), (a'), (a'') etc. are difficult to reconcile with the experimental observations since they would involve the emission of several energetic nucleons with a high probability for the observation of other tracks at point Y.

On the other hand, reaction mechanisms such as (b) and (b') corresponding to a π-meson primary, are in very good agreement with the observations if it be assumed that the recoil particle W is either a single nucleon or neutral hyperon. The first reaction is the most compatible with the measured value of $p\beta$ for particle A: the second however cannot be excluded because of the errors in the estimation of the multiple scattering and because of the arbitrary assumption that the target nucleon was at rest. An alternative hypothesis consistent with the kinematical analysis is that of the emission of an excited di-neutron ($\Lambda^0 + n$) with the two K-mesons. Such an interpretation depends of course upon the existence of such a particle — the experimental evidence at present available makes it appear unlikely: in fact assuming the force between a Λ^0-particle and a nucleon to be charge independent, the binding energy of such a particle should be equal to that of the excited deuteron and there is yet no clear evidence for such a bound system.

For obvious reasons reactions of types (a''') and (b''') have not been discussed in detail, and also those in which a Σ-particle is produced since the dynamics will be very similar to those of (a') and (b'). The production of a Ξ-particle is much less likely to have occurred than the emission of a Λ^0-particle, but one cannot exclude absolutely this possibility. However, reaction (b'') involving the production of a recoil composed of a 0^0-particle and a nucleon appears to be in disagreement with the observed kinematical configuration.

Reactions of type (b), (b') where the primary particle was a K^--meson instead of a π-meson would appear to agree well on dynamical grounds with the observations. We have not considered these further however, because

of the direct measurements on the mass of the particle A. In view of the complete absence of information on the inelastic interactions of very energetic negative heavy mesons, it appears unfair to base a probability argument upon the fact that due to their greater abundance, π-meson induced reactions will be more frequent than those produced by K^--mesons.

Reactions in which two or more neutral particles are produced have an infinite number of dynamical solutions and so cannot be treated satisfactorily. It suffices to point out that those cases corresponding to the emission of the recoiling particles as a unique aggregate W, imply the minimum energy for particle A provided the other conditions remain the same. For example, the reaction:

$$N + N = 2K + 2N ,$$

which is in disagreement with the experimental data when the two neutrons are emitted as a single body, becomes even more incompatible if the two nucleons take different directions.

The present discussion has been based entirely upon the hypothesis that the target nucleon was at rest and so implies that it was a hydrogen nucleus rather than a nucleon bound in a nucleus. But in fact the conclusions concerning a $N + N$ interaction remain unaltered even if the target nucleon is allowed the maximum possible Fermi momentum and therefore the discussion is still approximately valid for these reactions. However motion of the target nucleon will smooth considerably the differences among the various reactions resulting from a $\pi + N$ collision. Taking into account that the present event is the only example observed of a K^--meson being produced from a collision which might have involved a hydrogen nucleus in a total of about 40 cases, it is not unreasonable to suppose that the parent nucleus was indeed that of hydrogen since about one nuclear interaction in twenty in emulsion takes place with hydrogen.

4. – Conclusions.

We may conclude with the following remarks:

(i) The event Y has a high probability of being due to the pair production of K-mesons. That the two heavy mesons were produced in independent processes appears unlikely in view of the apparent simplicity of the event.

(ii) Of the various reaction schemes whereby one may interpret the event, that which agrees best with the experimental data is the following:

$$\pi^- + p = K^+ + K^- + n .$$

It is worthy of note that this reaction, is one of the few, among all those previously discussed, which are allowed by the selection rules of Gell-Mann and Pais ([10]).

Other reactions in which a K^--meson replaces the π-meson or a Λ^0-particle the neutron, cannot be absolutely excluded but must be considered in less satisfactory agreement.

(iii) Having regard to the comparatively small number of negative K-mesons hitherto observed to be produced in emulsions it would appear that the associated production mechanism is a rather frequent one even at primary energies not far above the threshold value, and therefore constitutes a mechanism for the production of K^--mesons which, if not the exclusive one, is at least preferred. These considerations together with others concerning the frequencies of $K^+ + K^-$ compared to $K^+ + Y$ production events suggested by the Bristol Group ([9]) seem to provide further support for some of the selection rules for K-meson production suggested by GELL-MANN and NISHIJIMA ([10]).

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We wish to thank Professors A. ROSTAGNI, N. DALLAPORTA, G. OCCHIALINI and G. PUPPI for useful suggestions and discussions, and dott. D. KEEFE for reading and translating the manuscript.

We wish also to thank the observers A. BERNARDI, V. CHIARATTI and G. MARINOLLI for help with the measurements.

Event X was observed by Mr. G. CALTABIANO and event Y by Mr. V. CHIARATTI.

([9]) BRISTOL GROUP: private communication.

([10]) M. GELL-MANN: *Reports of the Pisa Conference*; K. NISHIJIMA: *Progr. Theor. Phys.*, **12**, 107 (1954).

RIASSUNTO

Si descrive un evento nel quale una particella energica con carica unitaria produce una stella dalla quale i soli prodotti di disintegrazione carichi uscenti sono un mesone K negativo ed un'altra particella di circa 1000 masse elettroniche. L'interesse dell'evento sta nel fatto che esso è il primo caso in cui si può specificare un meccanismo di creazione del mesone K negativo con un notevole grado di attendibilità. L'evento risulta in accordo colle regole di selezione derivanti dalle idee relative alla produzione associata.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Angular Correlation in Cascade Decay.

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(ricevuto il 25 Giugno 1955)

Examples of cascade decays of the kind

$$\Xi^- \to \Lambda^0 + \pi^-, \qquad \Lambda^0 \to \mathrm{p} + \pi^-, \tag{1}$$

have been recently reported by different authors [1]. The study of the possible angular correlations in the cascade decay (1) could furnish interesting conclusions on the spins and parities of the Ξ^- and of the Λ^0.

Let us consider a free Ξ^- in its center of mass system (fig. 1), decaying with the emission of a Λ^0 and a π^-. The Λ^0 undergoes decay into a proton and a π^-. Let θ be the angle between the direction of motion of the Λ^0 and the direction of motion of one of its decay products in the center of mass system. To indicate the case referring to a Ξ^- with spin S and parity P (relative to the proton) and to a Λ^0 with spin s and parity p (relative to the proton) we write $[SP_{sp}]$. In the initial state (Ξ^- at rest) the coordinate system is chosen without reference to any privileged direction, and the statistical ensemble will be described by a density matrix

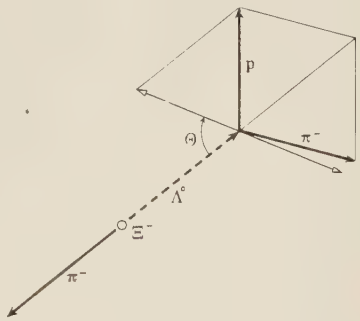

Fig. 1.

$$\langle Sm_S | \varrho(i) | Sm_S' \rangle = \delta m_S,\, m_S' .$$

For the derivation of the angular correlation we may assume that perturbation theory is applicable, since the two decay processes (1) occur in very long time and they

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(1) R. Armenteros, K. H. Barker, C. C. Butler, A. Cachon and C. M. York: *Phil. Mag.*, **43**, 597 (1952); C. M. York, R. B. Leighton and E. K. Bjornerud: *Phys. Rev.*, **90**, 167 (1953); C. D. Anderson, E. W. Cowan, R. B. Leighton, V. A. J. van Lint: *Phys. Rev.*, **92**, 1089 (1953); E. W. Cowan: *Phys. Rev.*, **94**, 161 (1954); W. B. Fretter and W. Friessen: *Bull. Am. Phys. Soc.*, **29**, 9 (1954); W. H. Arnold, J. Ballam, G. K. Lindenberg and V. A. J. van Lint: *Phys. Rev.*, **98**, 838 (1955), see also *Proceedings of the Rochester Conference* 1955.

should be due to weak interactions. We indicate with l_1, l_1' the orbital angular momenta relative to center of mass system, between the Λ^0 and the π^-, when the Ξ^- disintegrates. Parity conservation requires that the possible values of l_1, l_1' differ by even integers. Similarly, parity conservation requires that, for a given s and p, the proton and the π^-, when the Λ^0 disintegrates, are emitted with a unique value of their relative orbital angular momentum, which we call l_2. The angular correlation is given, in the general case, by

$$(2) \qquad F(\cos\theta) = \sum_{\nu, l_i, l_i'} T(l_1) T(l_1') (l_1 l_1' 00 \,|\, \nu 0)(l_2 l_2 00 \,|\, \nu 0) W(ss l_1 l_1' \nu S) W(ss l_2 l_2 \nu \tfrac{1}{2}) P_\nu(\cos\theta) .$$

In the above formula $T(l)$ is related to the reduced matrix element $(S\|l\|s)$ by the relation $T(l) = (2l+1)^{\frac{1}{2}}(S\|l\|s)$. The coefficients $(ll'00\,|\,\nu 0)$ are particular Clebsch-Gordan coefficients and $W(ssll'\nu S)$ are particular Racah coefficients. Of course, only even values of ν contribute to the sum.

In the cases $[SP\tfrac{1}{2}p]$ the angular correlation is always isotropic, as must be expected on the basis of very general theorems, while for $[SP\tfrac{3}{2}p]$ it has the form

$$F(\cos\theta) = 1 + a\cos^2\theta ,$$

and for $[SP\tfrac{5}{2}p]$ it has the form

$$F(\cos\theta) = 1 + a\cos^2\theta + b\cos^4\theta ,$$

and so on. The values of the coefficients may easily be derived from (2). Let us consider the cases $[SP\tfrac{3}{2}p]$. The values of l_1, l_1' and of l_2 are given in the table I. The values of $[SP]$ are indicated in the first column, the values of $[sp]$ referring to each column are given in the first row. The angular correlations which are found in these cases are

TABLE I.

	$[\frac{3}{2}+]$ $(l_2=1)$	$[\frac{3}{2}-]$ $(l_2=2)$
$[\frac{1}{2}+]$	1	2
$[\frac{1}{2}-]$	2	1
$[\frac{3}{2}+]$	1, 3	0, 2
$[\frac{3}{2}-]$	0, 2	1, 3
$[\frac{5}{2}+]$	1, 3	2, 4
$[\frac{5}{2}-]$	2, 4	1, 3
. . . .	. . . . .	. . . . .

$[\frac{1}{2}+\frac{3}{2}+], [\frac{1}{2}-\frac{3}{2}-]$ $F = 1 + 3\cos^2\theta$

$[\frac{1}{2}-\frac{3}{2}+], [\frac{1}{2}+\frac{3}{2}-]$ $F = 1 + 3\cos^2\theta$

$[\frac{3}{2}+\frac{3}{2}+], [\frac{3}{2}-\frac{3}{2}-]$ $F = 1 - 0{,}857\cos^2\theta + \xi(0{,}280 - 0{,}842\cos^2\theta) + \xi^2(0{,}184 + 0{,}367\cos^2\theta)$

$[\frac{3}{2}-\frac{3}{2}+], [\frac{3}{2}+\frac{3}{2}-]$ $F = 1 + \xi(0{,}447 - 1{,}342\cos^2\theta) + 0{,}200\xi^2$

$[\frac{5}{2}+\frac{3}{2}+], [\frac{5}{2}-\frac{3}{2}-]$ $F = 1 + 3{,}333\cos^2\theta + \xi(0{,}713 - 2{,}138\cos^2\theta) + \xi^2(0{,}429 - 0{,}143\cos^2\theta)$

$[\frac{5}{2}-\frac{3}{2}+], [\frac{5}{2}+\frac{3}{2}-]$ $F = 1 - 0{,}789\cos^2\theta + \xi(0{,}384 - 1{,}153\cos^2\theta) + \xi^2(0{,}263 + 0{,}438\cos^2\theta)$

We have introduced here the parameter ξ which is a real number and is given by

$$\xi = \frac{T(l_1 + 2)}{T(l_1)},$$

where l_1 is the first of the two values reported for each case of table I. As one sees, ξ is the only physical parameter entering in the angular correlation, but it is impossible to predict its exact value in the absence of a complete theory. One can make an approximate estimate, assuming that the interaction has a small definite range. In that case one would expect that ξ has a small value, and, as a first approximation, one can try to assume that all terms proportional to ξ and ξ^2 are negligible in the angular correlation formula. We note that all cases $[\frac{1}{2}P\frac{3}{2}p]$ give the same correlation, that is, no conclusion on the relative parity of the Ξ^- and of the Λ^0 could be obtained from the measurement. If ξ is small, as expected, the correlation in the cases $[\frac{3}{2}- \frac{3}{2}+]$ and $[\frac{3}{2}+ \frac{3}{2}-]$ is nearly isotropic. A similar situation, of course, also occurs in the cases $[\frac{5}{2}- \frac{3}{2}+]$ and $[\frac{5}{2}+ \frac{5}{2}-]$, and similarly for larger spins.

As already mentioned, the angular correlation is exactly isotropic if the Λ^0 has spin $\frac{1}{2}$, corresponding to the general theorem that particles with spin $\frac{1}{2}$ always decay isotropically. A simple proof can be given as follows. Let $f(\mu_+)$ be the number of decays with direction of emission $\boldsymbol{n}$ for the spin up situation (fig. 2*a*). For the spin down situation (fig. 2*b*) the number of decays with direction of emission $\boldsymbol{n}$ will be $f(\mu_-) = f(-\mu_+)$. But parity conservation requires $f(-\mu) = f(\mu)$, and, since $f(\mu_+)+f(\mu_-)$ must be isotropic, it follows that $f(\mu_+)$ must also be isotropic.

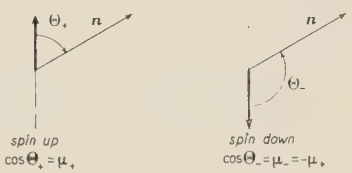

Fig. 2.

As it has already been pointed out by Treiman, Reynolds and Hodson [2], any departure from isotropy in the two body decay of an unstable particle necessarily implies a spin higher than $\frac{1}{2}$ (the reverse, of course, is not true). This remark should be useful to recognize if particles with spin higher than $\frac{1}{2}$ exist. To get more detailed information on the spins and on the parities, we believe that the case of the cascade decays discussed here presents some advantages with respect to other cases which involve the knowledge of the matrix elements at the production. In the present case a smaller number of matrix elements intervenes in the angular correlation formula, and, what seems more important, their ratios are fixed numbers which do not depend on energy. It is expected that only the lowest orbital angular momentum is important in determining the correlation, although no rigorous proof can be given.

With regard to the general question whether possible high spins of the new particles are alone sufficient to give an explanation of the many peculiar phenomena observed (such as long lifetimes, associated production, particular absorption processes etc.) it seems at present that such a viewpoint would be faced by many difficulties. Particular arguments have been given by different authors [3] and

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[2] S. B. Treiman, G. T. Reynolds and A. L. Hodgson: *Phys. Rev.*, **97**, 244 (1955).

[3] M. Gell-Mann and A. Pais: *Proceedings of the Glasgow Conference*, 1954; R. Gatto: *Nuovo Cimento*, **1**, 372 (1955); R. H. Dalitz: preprint.

others could be developed. However it cannot be excluded that some of the new particles may have spin higher than 0 or $\frac{1}{2}$.

* * *

I would like to take the opportunity to thank my colleagues of the emulsion group of Rome University for stimulating conversations. Moreover I would like to thank Prof. R. W. THOMPSON for an useful discussion.

On Pion-Nucleon Scattering.

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(ricevuto il 9 Agosto 1955)

The analytical method for the calculation of the pion-nucleon phaseshifts, outlined in a previous paper [1], has been reformulated for negative elastic and charge-exchange scattering. By means of the nine equations relating the experimental quantities $A_\pm$, $B_\pm$, $C_\pm$, A_0, B_0 and C_0 to $\boldsymbol{a}_\pm$, $\boldsymbol{b}_\pm$, $\boldsymbol{c}_\pm$, $\boldsymbol{a}_0$, $\boldsymbol{b}_0$ and $\boldsymbol{c}_0$, it is found that α_{11} and α_{13} can be expressed as functions of α_1 through the following relations

$$\cos 2\alpha_{13}^{(\pm)} = \lambda \pm (\lambda^2 - \mu)^{\frac{1}{2}}, \qquad \cos 2\alpha_{11}^{(\pm)} = p - 2\cos 2\alpha_{13}^{(\pm)}, \tag{1}$$

where

$$\lambda(\alpha_1) = p(2+q)(5+4q)^{-1}, \qquad \mu(\alpha_1) = (p^2+q^2-1)(5+4q)^{-1}, \tag{2}$$

$$p(\alpha_1) = 3 + (r/2) - 2(A + C/3), \qquad q(\alpha_1) = 1 + (r/2) - 2A, \tag{3}$$

$$r(\alpha_1) = 2(1 - \cos 2\alpha_1), \tag{4}$$

$$\begin{cases} A = (1/4)(|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2) = (1/2)[3(A_- + A_0) - A_+], & \boldsymbol{a} = (1/2)(\boldsymbol{a}_- - \boldsymbol{a}_+), \\ B = (1/4)(\boldsymbol{a}^*\boldsymbol{b} + \boldsymbol{a}\boldsymbol{b}^*) = (1/2)[3(B_- + B_0) - B_+], & \boldsymbol{b} = (1/2)(\boldsymbol{b}_- - \boldsymbol{b}_+), \\ C = (1/4)(|\boldsymbol{b}|^2 - |\boldsymbol{c}|^2) = (1/2)[3(C_- + C_0) - C_+], & \boldsymbol{c} = (1/2)(\boldsymbol{c}_- - \boldsymbol{c}_+). \end{cases} \tag{5}$$

Eqs. (1)-(5), which show that the phaseshifts for isotopic spin $\frac{1}{2}$ do not depend explicitely on those for isotopic spin $\frac{3}{2}$, are formally identical with Eqs. (1)-(5) of I, valid for positive scattering [2], provided α_{33}, α_{31}, α_3 are substituted by α_{13}, α_{11}, α_1,

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[1] E. Clementel, G. Poiani and C. Villi: *Nuovo Cimento*, **2**, 389 (1955). This paper will be quoted here as I and the same notation will be used.

[2] In terms of the quantities A, B and C, i.e. of *all* the nine experimental coefficients, the determination of phaseshifts for $T = \frac{1}{2}$ state could therefore be carried out using the Ashkin and Vosko graphical method for positive scattering [*Phys. Rev.*, **91**, 1248 (1953)].

and the quantities A_+, B_+, C_+ by the quantities A, B, C, defined in Eqs. (5). It follows that the $T = \frac{1}{2}$, P-phaseshifts *must* satisfy at any energy the condition $\alpha_{13}^{(+)} - \alpha_{11}^{(+)} = \alpha_{11}^{(-)} - \alpha_{13}^{(-)}$. For this reason, we shall call conventionally Fermi and Yang solutions for isotopic spin $\frac{1}{2}$ those labelled with (—) and respectively with (+). The phaseshift α_1 is fixed by solving the equation $4B = f(\alpha_1)$, where

$$f(\alpha_1) = r(3 - p) + 2 \sin 2\alpha_1 (2 \sin 2\alpha_{13}^{(\pm)} + \sin 2\alpha_{11}^{(\pm)}) . \tag{6}$$

Spurious solutions, arising from the fourfold ambiguity introduced by the procedure ($\sin 2\alpha_{11} \gtrless 0$, $\sin 2\alpha_{13} \gtrless 0$), are eliminated taking into account the definition of A in terms of the phaseshifts, i.e. $4A = 2 + r - 2 \cos 2(\alpha_{13} - \alpha_{11})$.

The mathematical symmetry between the relations defining the phaseshifts for $T = \frac{1}{2}$ and $T = \frac{3}{2}$ states has made possibile to build up a mechanical analyzer, which performs automatically all operations required for the six phaseshift determination. Resorting to the mouvement in a scattering analysis has the great advantage to visualize the mutual dependence of the phaseshifts in fitting the experimental data. The analyzer was found also particularly useful to test the phaseshift stability against variations of the nine experimental coefficients.

Since the mechanical behavior of the analyzer, i.e. its answer to a given set of input coefficients A, B and C, must be ruled by Eqs. (1)-(5), one has to carry out the kinematical translation of the following pion-nucleon phaseshift equations [3]

$$\begin{cases} \cos 2\alpha_i + 2 \cos 2\alpha_{i3} + \cos 2\alpha_{i1} = u_i , \\ \sin 2\alpha_i + 2 \sin 2\alpha_{i3} + \sin 2\alpha_{i1} = v_i , \\ \cos 2\alpha_i + \cos 2(\alpha_{i3} - \alpha_{i1}) = w_i , \end{cases} \tag{7}$$

where $i = 2T$ is twice the total isotopic spin and

$$\begin{cases} u_i = 2 + w_i - (2/3)C_i , \\ v_i = \pm [2(u_i + 2w_i + 2B_i) - u_i^2]^{\frac{1}{2}} , \\ w_i = 2(1 - A_i) . \end{cases} \tag{8}$$

The input coefficients A_i, B_i and C_i are of course to be identified with A_+, B_+ and C_+ for $i = 3$, and with A, B and C, given by Eqs. (5), for $i = 1$.

The working principle of the phaseshift analyzer is given in Fig. 1. The articulated system $[OQ_1Q_2P_1Q_3]$ is fixed at the origin O, and the arm OQ_1 is free to rotate around O in the plane xy. The point P_1 is kept fixed, for a given pion energy, during all operations. Its coordinates are chosen in such a way that $OP_1' = u_i$ and $OP_1'' = v_i$. We identify α_i with the only degree of freedom of the system, and regard therefore the other two phaseshifts as functions of α_i. When α_i is varied

[3] Only the first and the third of Eqs. (7) are directly given by the scattering theory; from these equations the second one can be derived making use of the following relation

$$2 \cos 2(\alpha_{i3} - \alpha_i) + \cos 2(\alpha_{i1} - \alpha_i) + 2 \cos 2(\alpha_{i3} - \alpha_{i1}) = 5 - 6A_i + 2B_i - (2/3)C_i .$$

between the upper and lower limit, dependent on the pion energy ([4]), the system undergoes a continous deformation, because the arms $P_1Q_2 = P_1Q_3 = 1$ and $Q_1Q_2 = Q_1Q_3 = 2$ are allowed to rotate around the junctions at Q_1, Q_2, Q_3 and P_1. This device realizes mechanically the parametrization of α_{i3} and α_{i1} as functions of α_i, and performs kinematically all computations involved in Eqs. (1)-(5) for $i = 1$, and in Eqs. (1)-(5) of I for $i = 3$. The first two of Eqs. (7) are thus satisfied, but, among the continuity of solutions obtainable by varying α_i, one must select those

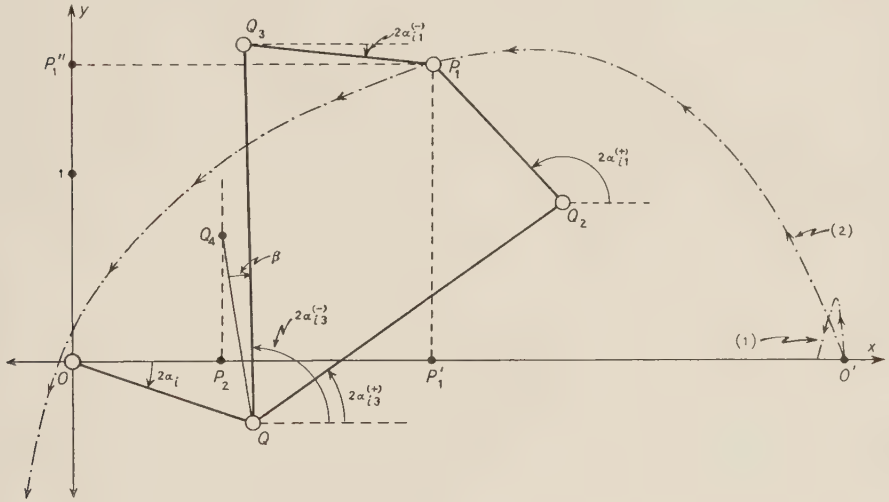

Fig. 1. – Phaseshift analyzer for pion-nucleon scattering. Solutions labelled with (+) and (—) are Yang and Fermi solutions for isotopic spin $T = i/2$ $(i = 1, 3)$. Curve (1) and (2) show the successive positions of the point P_1 for negative and respectively for positive scattering, from zero pion energy to about 210 MeV [see footnote ([11])].

which satisfy the third of Eqs. (7) either. Therefore, a new limitation must be imposed to the mouvement of the system. To realize this in the simplest, but not unique, way, we take profit of the arm $Q_1Q_4 = 1$, which is allowed to rotate around the junction at Q_1, while its end-point Q_4 lies on the parallel to the y-axis at the distance $OP_2 = w_i$. Once the points P_1 and P_2 are fixed for the considered scattering energy, the system has to be deformed by varying α_i until the angle β between the arms Q_1Q_4 and Q_1Q_3 becomes equal to $2\alpha_{i1}^{(-)}$. When this condition is realized, the analyzer is in the resolution position, and the phaseshifts are obtained by direct reading.

Eqs. (8) show how the points P_1 and P_2 must be moved, for a given pion energy, in order to test the phaseshift stability against variations of the input coefficients A_i, B_i and C_i. It is seen that the simplest test is that concerning the stability against variations of B_i, since in this case the only operation required is to move the point P_1 parallel to the y-axis, keeping fixed the point P_2. This corresponds to mathematical situation envisaged by Eq. (6). Furthermore, if also P_2 is moved along the x-axis, the previous operation permits to test, for a given B_i, the stability

([4]) The mathematical condition is given by $\cos 2\alpha_i \geq 1 - 2A_i$. For the upper and lower values of α_i, the points Q_2 and Q_3 coincide, making Fermi and Yang solutions identical, as it has been found analyrically in ref. ([1]).

against a simultaneous variation of A_i and C_i, consistent with the experimental value of the total cross-section. The independent variation of A_i or C_i requires a combined mouvement of the point P_1 parallel to both the x- and y-axis, the only operational difference being that the variation of A_i implies also the mouvement of the point P_2.

The sign ambiguity for v_i is a consequence of the well known fact that the set of phaseshifts with all signs reversed is still a solution of the scattering problem. Therefore, the analyzer has two resolution positions corresponding to the point P_1 and to its symmetric respect to the x-axis. It follows that the *a priori* total number of sets is eight, i.e. two for each of the four cases $[\alpha_i \gtrless 0,\ v_i \gtrless 0]$. It is easy, however, to establish how to handle the analyzer in order to obtain an answer consistent with the causality prescriptions of the scattering theory ([5]), since the dependence of the real part of the forward scattering amplitude $f_i(k)$ on the input coefficients is given, in the S and P wave approximation, by the quantity v_i according to the relation

$$v_i(k) = 2kRef_i(k)\,, \tag{9}$$

where

$$f_1(k) = (3/2)f_+(k) - (1/2)f_-(k), \qquad f_3(k) = f_+(k)\,. \tag{10}$$

Causality conditions require ([6]) now $Ref_3(k)$ to be a positive function of the energy changing its sign near 180 MeV. As a consequence, v_3 is positive up to the energy satisfying the condition $v_3(k) = 0$, for which the second of Eqs. (7) gives

$$\frac{\mathrm{d}\sigma_3(0^\circ)}{\mathrm{d}\Omega} = \left(\frac{k}{4\pi}\right)^2 [\sigma_3(k)]^2\,, \tag{11}$$

where $\sigma_3(k)$ is the total cross-section for positive scattering. This result is readily obtainable also as a particular case $[Ref_3(k) = 0]$ of very general considerations using the optical theorem. With this requirement for the sign of v_3, it can be seen that the analyzer excludes positive values of α_3 for pion energies higher than that satisfying the relation ([11]) $B_+ = 2(A_+ - 1) = -w_3$. Being A_+ always positive and higher than unity only when $\alpha_{33} - \alpha_{31} \sim 90^\circ$, it follows that this energy is or is near to the inversion energy of the coefficient B_+. For lower energies, the negative value for α_3 is required by the continuity of all the phaseshifts.

According to the results of the phaseshift analysis for positive scattering ([1]), the inequality $\alpha_{33}^{(+)} < \alpha_{33}^{(-)}\ [\alpha_{33}^{(+)} - \alpha_{31}^{(+)} = \alpha_{31}^{(-)} - \alpha_{33}^{(-)}]$, valid at 120 and 135 MeV, becomes $\alpha_{33}^{(+)} > \alpha_{33}^{(-)}\ [\alpha_{33}^{(+)} - \alpha_{31}^{(+)} = \pi - (\alpha_{33}^{(-)} - \alpha_{31}^{(-)})]$ at 300 MeV. It follows that the crossing ([7]) of the curves expressing $\alpha_{33}^{(+)}$ and $\alpha_{33}^{(-)}$ as functions of the pion energy obviously implies that, at some energy lower than 300 MeV, Fermi and Yang solutions become

([5]) R. KARPLUS and M. A. RUDERMAN: *Phys. Rev.*, **93**, 771 (1955); M. GOLDBERGER, H. MIYAZAWA and R. OEHME: in press. We thank Dr. R. OEHME for making available to us this paper prior to publication.

([6]) H. L. ANDERSON, W. C. DAVIDON and U. E. CRUSE: in press. We thank Prof. ANDERSON for the opportunity to read this paper prior to publication.

([7]) See in this connection section 8 of the paper: F. DE HOFFMANN *et al.*: *Phys. Rev.*, **95**, 1586 (1954).

identical, and the energy itself is fixed according to the equation $\lambda_+^2(k) = \mu_+(k)$ [1,11]. This is the circumstance that, together with the continuity of all phaseshifts versus pion energy, insures the smallness of the Fermi $\alpha_{31}^{(-)}$ phaseshift throughout the energy region, as it is beautifully presented to view by the phaseshift analyzer. If this were not the case, i.e. if $\alpha_{33}^{(+)}$ would not increase faster than $\alpha_{33}^{(-)}$, the phaseshift $\alpha_{31}^{(-)}$ would exhibit an anomalous behavior, going from negative values to positive ones just before the $\alpha_{33}^{(-)}$-resonance, and increasing afterwards with the energy. This behavior would be conflicting with field theoretical calculations, based on the hypothesis of an extended source and linear coupling [8], leading to an α_{31} negative and equal to α_{11}, which is certainly small.

For negative scattering the situation is somewhat less clear, mainly because the experimental uncertainties, large compared to the values of A, B and C, make the determination of the phaseshifts for isotopic spin $\frac{1}{2}$ of course rather poor. Since A, B and C have order of magnitude 10^{-2}, the analyzer is nearly stretched along the x-axis. This typical position visualizes the fact that α_{11} and α_{13} are small [9], and shows that Fermi and Yang solutions for $T = \frac{1}{2}$ state should not sensibly differ. This circumstance probably explains why MANIAC calculations, for instance at 120 MeV [10], have shown that only two distinct sets of six phaseshifts are compatible with the experimental data. It is, in fact, clear that *at any energy four and only four distinct sets of six phaseshifts are consistent with given values of* A_i, B_i *and* C_i. These sets are formed combining each of the two solutions for $T = \frac{3}{2}$ state with the two valid for $T = \frac{1}{2}$ state, and selection of the correct one should in principle be made possible by the polarization of the recoil nucleon. To emphasize the spectacular difference in the determination of the phaseshifts for $T = \frac{3}{2}$ and $T = \frac{1}{2}$ states, we have drawn in Fig. 1 the track of the point P_1 from zero pion energy ($00' = 4$) up to about 210 MeV for $i = 1$ (curve 1) and $i = 3$ (curve 2) [11].

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[8] See G. F. CHEW and F. E. LOW's communications presented at the *Rochester Conference*, 1955, and related papers on this subject. The equality $\alpha_{31} = \alpha_{11}$, following from the still provisional form of the cut-off theory, is not necessarily required by the scattering relations.

[9] If one assumes $\alpha_{11} = \alpha_{13} = 0$, from the third of Eqs. (7) one gets $\cos 2\alpha_1 = 1 - 2A$, which corresponds to Eq. (6) used in a previous note [E. CLEMENTEL, G. POIANI and C. VILLI: *Nuovo Cimento*, **2**, 352 (1955)].

[10] H. A. BETHE and F. DE HOFFMANN: *Mesons and Felds* (New-York, 1955), p. 79.

[11] The illustrative curves (1) and (2) have been calculated using the coefficients given in Table XVI of ref. [7]. With these coefficients it is found that the analyzer excludes positive values of α_3 for energies higher than $\sim$ 170 MeV. Assuming the following linear dependence $\alpha_3 = 9.5° - 17.1°\eta$ of α_3 on the relative momentum η valid in the energy region where the equation $\lambda_+^2(k) = \mu_+(k)$ is expected to hold, the crossing point of $\alpha_{33}^{(+)}$ and $\alpha_{33}^{(-)}$ is found to be somewhere between 170 and 180 MeV.

Axially Symmetric Solution in Problems of Galactic Magnetic Fields and a New Type of Red Shift.

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(ricevuto il 17 Agosto 1955)

The hypothetical case of a galactic magnetic field due to a monopole has already been treated [1,2] with sperically symmetric solutions. As only dipoles have a physical existence, it may be of interest to treat this case which then needs the application of axially symmetric solution.

External solution. In the galactic space, we put $J^\mu = 0$ and

$$F_{\mu\nu} = \begin{Vmatrix} 0 & H_z & -H_y \\ -H_z & 0 & H_x \\ H_y & -H_x & 0 \end{Vmatrix} .$$

We then have the the following conditions:

(1) $$\partial F_{12}/\partial x_2 = 0 ,$$

(2) $$\partial F_{23}/\partial x_3 = 0 ,$$

(3) $$\partial F_{31}/\partial x_1 = 0 ,$$

and

(4) $$\frac{\partial F_{\mu\nu}}{\partial x^\delta} + \frac{\partial F_{\nu\delta}}{\partial x^\mu} + \frac{\partial F_{\delta\mu}}{\partial x^\nu} = 0 ,$$

which now reduces into div $B = 0$.

(*) Present address: Forschungsprofessur Prof. Dr. Jordan, Universität Hamburg.

(1) R. L. Brahmachary: *Nuovo Cimento*, **1**, 953 (1955).

(2) R. L. Brahmachary: *Nuovo Cimento*, **1**, 953 (1955).

Following WEYL [3] and DUTTA-MAJUMDAR [4], we adopt the following scheme for

$$g_{\mu\nu} = \begin{Bmatrix} -e^u & 0 & 0 & 0 \\ 0 & -e^u & 0 & 0 \\ 0 & 0 & -r^2 e^{-\omega} & 0 \\ 0 & 0 & 0 & e^{\omega} \end{Bmatrix}.$$

We thus calculate $\mathcal{J}^{12} = \sqrt{-g}F^{12} = g^{11}g^{22}F_{12} = e^{-u}rk_1'$ where

$$k_1' = \frac{\partial \Phi_1}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_1} = F_{12}\,.$$

Hence, from the conditions (1), (2), (3) we can deduce

$$\begin{aligned} k_1' &= C_1 e^u/p; & \partial C_1/\partial r &= 0\,, \\ k_2' &= C_2 e^{-\omega}; & \partial C_2/\partial \theta &= 0\,, \\ k_3' &= C_3 e^{-\omega}; & \partial C_3/\partial z &= 0\,. \end{aligned}$$

Again, condition (4) reduces to $\mathrm{d}/\mathrm{d}z(e^{-\omega}C_2) + \mathrm{d}/\mathrm{d}r(e^{-\omega}C_3) + \mathrm{d}/\mathrm{d}\theta(e^u C_1/r) = 0$.

We now find

$$\begin{aligned} t_1^1 &= -F^{1\alpha}F_{1\alpha} + \tfrac{1}{4}\, g_1^1 F^{\alpha\beta}F_{\alpha\beta} \\ &= \frac{1}{2r^2}\,(-A + Be^{-u-\omega} - Ce^{-u-\omega})\,, \end{aligned}$$

and similarly

$$t_2^2 = \frac{1}{2r^2}\,(-A - Be^{-u-\omega} + Ce^{-u-\omega})\,,$$

$$t_3^3 = \frac{1}{2r^2}\,(\;\; A - Be^{-u-\omega} - Ce^{-u-\omega})\,,$$

$$t_4^4 = \frac{1}{2r^2}\,(\;\; A + Be^{-u-\omega} + Ce^{-u-\omega})\,.$$

The solution of WEYL [3] and also that of DUTTA-MAJUMDAR [4] are not appropriate for our case, for, certain stresses do not vanish. LEVI-CIVITA [5] has however given a cylindrically symmetric solution for a homogeneous magnetic field:

$$(5) \qquad \mathrm{d}s^2 = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2 + \frac{(x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2)^2}{a^2 - r^2} - \\ - (C_1 \exp[x_3/a] + C_2 \exp[-x_3/a])^2(\mathrm{d}x^4)^2\,,$$

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[3] H. WEYL: *Ann. der Phys.*, **54**, 117 (1917).
[4] S. DATTA-MAJUMDAR: *Phys. Rev.*, **72**, 390 (1947).
[5] W. PAULI: *Encyclo. Math. Wiss.* (*Relativitätstheorie*), 735 (1921).

where

$$r = \left((x_1)^2 + (x_2)^2\right)^{\frac{1}{2}},$$

and

$$c_1, c_2 = \text{const}, \qquad a = c^2/KF .$$

If $F = 1$ gauss, a is of the order of 10^{20}.

This solution seems to show, that any possible effect due to the magnetic field will be negligible. But there is another non-negligible effect due to $g_{44} = f(t)$ as deduced from the case of a magnetic pole in an expanding universe (1), (2). We will also make a quantitative correction of the result given there.

There we had $t_1^1 = C_1/r_4$ (for a monopole) and $T_1^1 = \ddot{g} + \frac{3}{4}(\dot{g})^2$. But as this latter expression is deduced from an isotropic co-ordinate system

$$e^{\mu}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) + e^{\nu}\,\mathrm{d}t^2,$$

we should abandon it and use $T_1^1 = 1/r^2 \quad 4/gr^2$ as obtained from the system used in our work

$$\mathrm{d}s^2 = -e^{\lambda}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2 + e^{\nu}\,\mathrm{d}t^2 .$$

Following the method given in (1) and taking into account the relation

$$e^{\lambda} = \mu^2/4 = g^2/4 ,$$

we now have

$$e^{\nu_E} = g_{44} = 1 + \frac{C_1 g^2}{4r^2} + C_2 g^2/4r , \tag{6}$$

where C_2 is a constant of integration. Following Reissner and Nordström, we may identify C_2 with the mass of a nebula. As, in these cases $\lambda = -\nu$ we have $g_{44} = f(t)$ when $g_{11} = f(t)$.

In case of our dipole, we find from the relation

$$K_1' = C_1 e^{u}/r; \qquad \partial C_1/\partial r = 0 ,$$

deduced from (1), the dependence of the magnetic force an time. K_1' could be constant if C_1 were $f(t)$ such that $C_1 = e^{-u} + \text{const}$. But as $K_1' = C_1/r$ if $e^{u} = 1$, — that is, in a stationary metric, C_1 cannot be $f(t)$ if the field is static. If K_1' changes with time, K_2', K_3' must also change in the same ratio, if the symmetric aspect is to persist with time. Thus we see from (2) and (3) that $e^{\omega} = f(t)$ if $\mathrm{d}(K_1')/\mathrm{d}t = \mathrm{d}(K_2')/\mathrm{d}t$.

In de Sitter space $g_{44} = f(r)$ leads to a red shift for a « vibrating system » at a distance in space. Similarly, $g_{44} = f(t)$ leads to a red-shift for such a system « at a distance in time », — i.e. for light emitted by a nebula at a distant past. Milkutat (6) had pointed out such a shift in an electrically charged, expanding

(6) E. Milkutat: unpublished communication.

universe ([7,8]). (That however does not occur in flat expanding space).

This effect, now deduced from the physical grounds of a galactic magnetic field, is also in a sense a generalisation of Riemann's concepts. RIEMANN had at first assumed that bodies have an existence independent of position (i.e. in space). But then he attained the concept that metric is determined by material distribution ([9]). Thus in actual world, congruence is not strictly preserved — in other words, bodies have an existence dependent on position (space). In the light of Einstein's theory we may generalise it thus — intervals are dependent on position in space time.

([7]) R. L. BRAHMACHARY: *Naturwiss.*, **40**, 41 (1953).
([8]) R. L. BRAHMACHARY: *Naturwiss.*, **40**, 313 (1953).
([9]) B. RIEMANN: *Ueber die Hypothesen, welche der Geometrie zugrunde liegen.*

Decay of ^{134}Cs.

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(ricevuto il 17 Settembre 1955)

The disintegration of $^{134}Cs \rightarrow {}^{134}Ba$ has been studied by several investigators ([1-11]) and, although there exists considerable agreement concerning the main β and γ ray components, the decay scheme is by no means certain. There remains moreover the question as to the total number and end point energies of the β-rays emitted in the disintegration. Under these circumstances a reinvestigation of the decay of ^{134}Cs seemed indicated. Below is a description of the experiment which has been carried out at our laboratory and a decay scheme is proposed which is compatible with the results.

The β-spectrum and the internal conversion electrons of ^{134}Cs were obtained with a double thin lens β-ray spectrometer. The detector was a conventional type Geiger-Müller counter with replaceable end window. Thin nylon films, 10-50 $\mu g/cm^2$ thick, were used as counter windows. The radioactive source material was cesiumchloride in hydrochloride acid solution obtained from Oak Ridge. Thin, uniform sources, (10-25 $\mu g/cm^2$), were deposited on nylon or tygon films by the usual evaporation methods and the films supported by a lucite cylinder the sides of which had been milled so as to leave only a narrow ring held by four thin strips. The resolution of the instrument was varied between 1.5% and 2.5% except in the investigation of the endpoint of the highest energy β-component when a resolution of 3.5% and sources up to 50 $\mu g/cm^2$ were used.

The γ-spectrum of ^{134}Cs was obtained with a NaI crystal scintillation spectrometer with a single channel pulse height analyzer.

([1]) K. Siegbahn and M. Deutsch: *Phys. Rev.*, **71**, 483 (1947).

([2]) K. Siegbahn and M. Deutsch: *Phys. Rev.*, **73**, 410 (1948).

([3]) L. G. Elliot and R. E. Bell: *Phys. Rev.*, **72**, 979 (1947).

([4]) J. L. Meem jr. and Maienschein: *Phys. Rev.*, **76**, 328 (1949).

([5]) M. A. Waggoner, M. L. Moon and A. Roberts: *Phys. Rev.*, **80**, 420 (1950).

([6]) J. M. Cork, J. M. Le Blanc, W. H. Nester, D. W. Martin and M. K. Brice: *Phys. Rev.*, **90**, 444 (1953).

([7]) D. C. Lu and M. L. Wiedenbeck: *Phys. Rev.*, **94**, 501 (1954).

([8]) M. C. Joshi and B. V. Thosor: *Phys. Rev.*, **96**, 1022 (1954).

([9]) G. L. Keister, E. B. Lee and F. H. Schmidt: *Phys. Rev.*, **97**, 451 (1955).

([10]) J. Verhaeghe and J. Demuynck: *Compt. Rend.*, no. 21, 1374 (1954).

([11]) G. Bertolini, M. Bettoni and E. Lazzarini: *Nuovo Cimento*, **1**, 746 (1955).

A Fermi plot of the β-spectrum of ^{134}Cs is shown in Fig. 1. An analysis of the β-spectrum reveals the presence of two principal β-groups with endpoint energies 80 ± 3 keV(28%) and 650 ± 5 keV (56%). The Fermi plot of both groups is straight, the log *ft* values are 6.3 and 9.0 respectively, and both transitions seem to be allowed in spite of the rather large *ft* value of the higher energy β-group. Near the endpoint of this group a change in slope occurs which, as KEISTER [9] has pointed out, is compatible with the presence of another β-group of endpoint energy 685 ± 10 keV (8%) and log *ft* value 9.9. As for the intermediate energy region there seems to be some question [6,9,11] as to whether it represents one or two β-components. Our experimental results are compatible with a resolution into two β-groups of maximum energies 210 ± 10 keV (3%) and 409 ± 4 keV (5%), in agreement with CORK [6]; the log *ft* values of the transitions are 8.6 and 9.3 respectively.

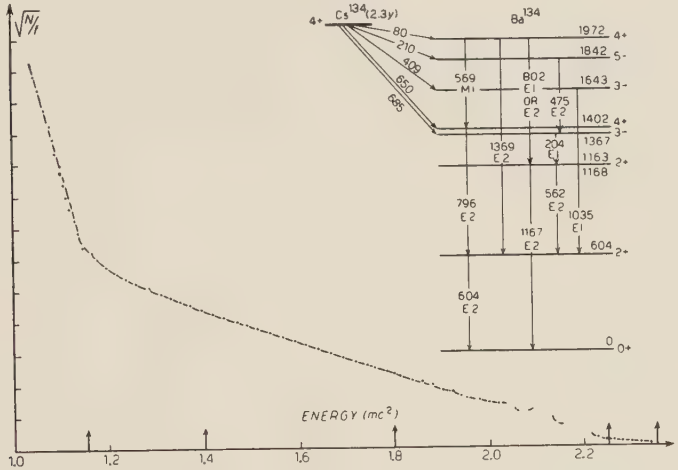

Fig. 1.

The γ-spectrum of ^{134}Cs obtained through internal and external conversion showed γ-rays of 204 ± 5, 475 ± 3, 562 ± 2, 569 ± 1, 604 ± 1, 796 ± 1, 802 ± 1, 1 035 ± 3, 1 167 ± 3 and 1 369 ± 3 keV. The relative intensities of the γ-rays in order of increasing energies are approximataly 0.1: 0.04: 0.14: 0.12: 1.00: 0.72: 0.11: 0.05: 0.03: 0.05. A weak γ-line was also observed at 658 ± 3 keV but, as it did not seem to fit into the decay scheme it was interpreted as due to some impurity, possibly ^{137}Cs.

The insert in fig. 1 represents the proposed decay scheme. It is essentially the decay scheme suggested by CORK [6] except that an energy level at 1 368 keV has been added to account for the presence of the fifth β-group. The order of the 475 keV and 204 keV γ-rays has been inverted and spin and parity assignments have been made for all energy levels.

^{134}Ba being an even-even nucleus, has a ground state of zero spin and even parity. The spin and parity of ^{134}Cs has been determined as 4 + ([12]). The 604 and 102 keV levels are 2 + and 4 + in agreement with the measured intensities and in accord with the results

([12]) E. H. BELLAMY and K. F. SMITH: *Phil. Mag.* (7), **44**, 33 (1953).

of other observers ([13,14]). Spin assignments of 2 + and 4 + for the 1 168 and 1 972 levels are compatible with the conversion coefficients for the 562, 1 167, and 802 keV γ-rays and seem to indicate 4 — 2 — 0 transitions for the major γ-cascades. Spins of 3 —, 3 —, and 5 — for the 1 367, 1 643, and 1 842 levels are consistent with the conversion coefficients of the remaining γ-rays and with the log *ft* values of the β-transitions.

The possibility of adjusting the decay scheme proposed by KEISTER ([9]) to our experimental data has been investigated. However, while neither of the two decay schemes can definitely be excluded, the decay scheme proposed above seems to be more consistent with our results.

([13]) B. L. ROBINSON and L. MASHONSKY: *Phys. Rev.*, **84**, 604 (1951).

([14]) D. G. ALKHAZOV, I. KH. LEMBERG and A. P. GRINBERG: *Izv. Akad. Nauk SSSR, Ser. Fiz.*, **17**, No. 4, 487 (1953).

Note on Hyperfragments.

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(ricevuto il 18 Agosto 1955)

Since the delayed disintegration of a heavy nuclear fragment with the release of a large amount of energy was first reported by Danysz and Pniewski [1], many events have been observed which establish the fact that a Λ^0 together with a number of nucleons can form systems which are stable for times of the order of the lifetime of the free Λ^0. If the strong interaction between the Λ^0 and the nucleon is attributed to the exchange of a heavy meson [2], the interaction can be expected to have a range of about one third and a depth about three times the range and depth, respectively, of the interaction between nucleons. With such an interaction neither the hyperdineutron ($n + \Lambda^0$) nor the hyperdeuteron ($p + \Lambda^0$), which form an i-spin doublet with spin singlet and triplet states, is stable [3]; and, in fact, there is no experimental evidence for such particles. We consider some of the simple consequences of the assumption of charge independent forces between two nucleons and charge symmetric forces between a Λ^0 and a nucleon.

The hypertrineutron ($^3n_\Lambda$), hypertritium ($^3H_\Lambda$), and hyperhelium-3 ($^3He_\Lambda$) form an i-spin singlet in spin doublet and quartet states, and an i-spin triplet in only spin doublet states, excluding orbital excitation. It seems reasonable to assume that the lowest lying state is a quartet (having three spins parallel) and hence is an i-spin singlet ($^3H_\Lambda$). There are three events in which the identity of the hypertriton has been definitely established [4]. The binding

(*) Work performed in the Ames Laboratory of the U. S. Atomic Energy Commission.

(1) M. Danysz and J. Pniewski: *Phil. Mag.*, **44**, 348 (1953).

(2) R. G. Sachs: *Phys. Rev.* (in press). See also A. Pais: *Phys. Rev.*, **86**, 663 (1952). The same coupling constant is assumed, in the present discussion, for the Λ^0-nucleon and nucleon-nucleon interactions; one may question, of course, whether this assumption is justified. Moreover, some admixture due to π-meson exchange can be expected.

(3) F. Duimio: *Nuovo Cimento*, **1**, 688 (1955).

(4) A. Bonetti, R. Levi Setti, M. Panetti, L. Scarsi and G. Tomasini: *Nuovo Cimento*, **11**, 210, 330 (1954). A. Debenedetti, C. M. Garelli, L. Tallone and M. Vigone: *Nuovo Cimento*, **12**, 466 (1954). M. Baldo, G. Belliboni, M. Ceccarelli, M. Grilli, B. Sechi, B. Vitale and G. T. Zorn: *Nuovo Cimento*, **1**, 1180 (1955). The binding energy of the Λ^0 as obtained by using the data of each of these three groups, is 0.7 ± 1 MeV, 0.3 ± 4 MeV, and -0.2 ± 1 MeV, respectively. H. Yagoda: *Phys. Rev.*, **98**, 153 (1955), has observed an event

energy of the Λ^0 is found to be less than one MeV. The small binding makes a second bound level unlikely; therefore, the hypertrineutron probably has no stable states; and, considering the additional coulomb energy, it is even less likely that the hyperhelium-3 is stable ([5]).

Of the four hypersystems with three nucleons and a Λ^0, hyperhydrogen-4 ($^4H_\Lambda$) and hyperhelium-4 ($^4He_\Lambda$) undoubtedly form a stable i-spin doublet differing in total binding by the coulomb energy. The binding energy of the Λ^0 can be expected to be very nearly the same in the two fragments. Observations indicate that the binding energy of the Λ^0 in $^4He_\Lambda$ is about 2.4 ± 1.5 MeV ([6]); evidence for the existence of $^4H_\Lambda$ is at present less convincing ([7]). Assuming that the $^4H_\Lambda$ is indeed stable, one concludes that the Pauli principle causes the forces between a third neutron and the nucleons of ^3H, which do not lead to a stable system, to be weaker than the forces between a Λ^0 and the nucleons of ^3H. However, the i-spin quartet of this group of hyperfragments is certainly unstable; for, neither ^4H nor ^4Li is stable, and the two extremes of the quartet are obtained by replacing the single proton or neutron which is unrestricted by the Pauli principle by a Λ^0, which is likewise unrestricted and for which the forces are somewhat weaker.

The hypersystems with four nucleons and one Λ^0 have a stable state which is an i-spin singlet, hyperhelium-5 ($^5He_\Lambda$). One example of this fragment has been reported ([8]). The two components of

which he interprets as being the disintegration of a hypertriton in which the Λ^0 is bound with 3.1 ± 1 MeV. One should note, however, that the event of this reference may possibly be an example of a $^4H_\Lambda$ rather than a $^3H_\Lambda$. The mode of disintegration could be either (*a*) $^4H_\Lambda \to {}^2H + p + n + \pi^- + \sim 30.9\pm1$ MeV, or (*b*) $^4H_\Lambda \to {}^3H + p + \pi^- + 31.0\pm1$ MeV. In both modes, the ^2H and ^3H are to be identified with the track which Yagoda designates by *A*. (The kinetic energy of the ^3H is about 2.01 ± 0.06 MeV.) The two possible modes lead to a binding of the Λ^0 in $^4H_\Lambda$ of about -0.3 ± 1 MeV and 5.9 ± 1 MeV, respectively.

([5]) An event, which may be the disintegration of either a $^3He_\Lambda$ or a $^4He_\Lambda$, has been observed by FRY, SCHNEPS and SWAMI: *The Disintegration of Hyperfragments* (*Univ. of Wis.*, preprint, Jan. 1955). Because the total binding energy found for $^3He_\Lambda$ (5.9±1 MeV) is almost twice that of $^3H_\Lambda$, the event is probably an example of a $^4He_\Lambda$.

If one assumes that the $^3n_\Lambda$ is indeed stable, one can examine its possible modes of disintegration. For example, the mode, (*a*) $^3n_\Lambda \to 2n + p + \pi^- + Q_a$, is especially interesting because the charged secondaries are the same as those of the decay of the free Λ^0. However, the apparent Q-value, i.e., the Q-value calculated from the observed kinetic energies of the p and π^-, would not have a unique value for all events of this mode of disintegration of the $^3n_\Lambda$. One may ask whether mode (*a*) could possibly account for some of the low, anomalous Q-values which have been observed for the decay of the free Λ^0. [See, for example, C. CASTAGNOLI, M. W. FRIEDLANDER, M. MERLIN and M. TEUCHER: *Suppl. Nuovo Cimento*, **12**, 460 (1954), especially p. 461]. For the mode, (*b*) $^3n_\Lambda \to t + \pi^- + Q_a$, both secondary particles are charged and could be observed; in this case the Q-value plus the binding energy of the Λ^0 would be 45.4 MeV.

([6]) HILL, SALANT, WIGDOFF, OSBORNE, PEVSNER, RITSON, CRUSSARD and WALKER: *Phys. Rev.*, **94**, 797 (1954), whose data give 3.1 MeV. NAUGLE, NEY, FREIER and CHESTON: *Phys. Rev.*, **96**, 1383 (1954), whose data give 2.7 ± 0.9 MeV. FRY, SCHNEPS and SWAMI: l. c., who find 3.9 ± 1 MeV. J. CRUSSARD and D. MORELLET: *Compt. Rend. Acad. Sci.* (Paris), **236**, 64 (1953); this event is interpreted as the disintegration of either a $^4He_\Lambda$ or a $^7Li_\Lambda$. (In each case, the Q-value is slightly ambiguous because the kinetic energy of the recoil nucleus (^3He or ^6Li) is not stated.) If the event is an example of a $^4He_\Lambda$, the binding energy of the Λ^0 ($\sim 9\pm4$ MeV) is somewhat higher than that of the other events quoted. See also BALDO *et al.*: l. c. The event of the latter reference is interpreted as the disintegration of either a $^4He_\Lambda$ or a $^6He_\Lambda$, with the binding energy of the Λ^0 equal to 1.7 ± 1.2 or 1.5 ± 1.2 MeV, respectively. B. STILLER, N. SEEMAN and M. M. SHAPIRO: *Bull. A.P.S.*, **30**, no. 5, 13 (1955), whose data give 1.9 ± 2.9 MeV. W. F. FRY: private communication, who finds 0.9 ± 0.5 MeV.

([7]) A preliminary estimate, based on a single event, reported in *Nature*, **175**, 972 (1955) gives a 1.2 ± 0.8 MeV for the binding energy of the Λ^0 in $^4H_\Lambda$. See also reference ([4]).

([8]) See reference ([7]). No estimate of the binding energy of the Λ^0 in the $^5He_\Lambda$ is given.

the i-spin triplet, $^5H_\Lambda$ and $^5Li_\Lambda$, are probably unstable because both ^{5}He and ^{5}Li are unstable. However, it is possible that the freedom in the orientation of the spins which exists in the two hypersystems and also the fact that $^4H_\Lambda$ and $^4He_\Lambda$ are less stable than ^{4}He might provide sufficient energy for binding. If the systems are indeed stable against nucleon emission, the binding energy of the Λ^0 in $^5H_\Lambda$ must be greater than the binding energy of the Λ^0 in $^4H_\Lambda$ and the binding energy of the Λ^0 in $^5Li_\Lambda$ must be greater than the binding energy of the Λ^0 in $^4He_\Lambda$.

These considerations can be extendep to hypersystems of greater mass number.

Current Density in Quantum Electrodynamics.

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(ricevuto il 30 Agosto 1955)

In previous papers a non linear operator equation for the current density $j(x)$ in Heisenberg representation has been derived [1]. This equation does not determine completely $j(x)$ but requires the previous knowledge of the matrix elements of j between states with no bosons.

The purpose of this letter is twofold. First we wish to give a derivation of the equation in which only the Heisenberg representation is used from the beginning. Secondly we shall discuss briefly the possibility of using of other finite relationships in order to obtain further information about the unknown part of j.

The Low equation has been a very useful tool for the investigation of the fixed source meson theory [3]. In this case, due to the absence of recoil, the determination of the no-meson part of $j(x)$ represents no problem. We believe, however, that a big advantage of this equation lies in the fact that, in relativistic theories with recoil, only the observed fermion mass appears everywhere.

On the other hand quantum electrodynamics is the only theory in which radiative corrections have been computed in a consistent way and successfully tested against experiment. For this reason we think useful to investigate separately the case of electrodynamics where gauge invariance has already allowed the derivation of other relationships between renormalized quantities (e.g. Ward's identities).

The Heisenberg operators $A_\nu(x)$ and $j_\nu(x)$ are connected by the well known equation

$$\Box A_\nu(x) = -j_\nu(x) \,. \tag{1}$$

As in LSZ we introduce the quantity

$$A_f(t) = i \int_{x_0=t} \mathrm{d}^3\boldsymbol{x} \left(A_\mu(x) \frac{\partial f_\mu(x)}{\partial x_0} - f_\mu(x) \frac{\partial A_\mu(x)}{\partial x_0} \right), \tag{2}$$

(1) S. Fubini: *Nuovo Cimento*, **2**, 180 (1955); M. Cini and S. Fubini: *Nuovo Cimento*, **2**, 192 (1955) denoted in the following as A and B. In these papers the connection with the results of F. Low (*Phys. Rev.*, **97**, 1992 (1955)) is discussed.

(2) H. Lehmann, K. Symanzik and W. Zimmerman: *Nuovo Cimento*, **1**, 205 (1955) denoted in the following as LSZ.

(3) G. F. Chew and F. Low: *Report at the Pisa Conference* (1955); and *Phys. Rev.* (to be published).

where $f_\mu(x)$ is any function of x satisfying the equation

$$\Box f_\mu(x) = 0 \,. \tag{3}$$

From eqs. (1), (2) and (3) one has

$$\frac{\mathrm{d}A_f(t)}{\mathrm{d}t} = i \int_{x_0=t} \mathrm{d}^3\boldsymbol{x} f_\mu(x) j_\mu(x) \,. \tag{4}$$

For $t \to -\infty$, $A_\nu(x)$ tends asymptotically to $A_\nu^{\mathrm{inc}}(x)$. As pointed out in LSZ $A_f^{\mathrm{inc}}(t)$ is time independent, and therefore it will be denoted as A_f^{inc}. Then eq. (4) can be integrated giving:

$$A_f(t) = A_f^{\mathrm{inc}} + i \int \mathrm{d}^4z \, f_\nu(z) \, j_\nu(z) \, \eta(t - z_0) \tag{5}$$

This equation coincides with eq. (B. 15) when $f_\nu(x)$ is taken as $-i\partial_{\nu\mu}D(x-y)$. It is easy to see that the quantity $A_f(t)$ coincides with our «generalized Heisenberg operator» $\varphi_t(y)$ and A_f^{inc} with $\varphi(y)$.

Since, as stated in B

$$[j_\mu(x), A_f(x_0)] = 0 \,, \tag{6}$$

we obtain

$$[j_\mu(x), A_f^{\mathrm{inc}}] = -i \int \mathrm{d}^4z [j_\mu(x), j_\nu(z)] f_\nu(z) \eta(x - z) \,. \tag{7}$$

Obviously eq. (7) coincides with (B. 11) with the preceding choice of $f_\nu(x)$ [4].

As pointed out in A and B, by expanding $j(x)$ in normal products of incoming operators $\psi^{\mathrm{inc}}(y_i)$ and $A^{\mathrm{inc}}(z_i)$ and taking into account the fact that any coefficient in this expansion is a function only of the differences $x - y_i$, $x - z_i$, one obtains the infinite set of Low equations. This property of the coefficients which was previously deduced from the equivalence principle [5], is, in Heisenbergr epresentation, a direct consequence of invariance of the equations connecting Heisenberg and incoming fields to four dimensional traslations. This fact has been simply stated in LSZ by introducing an operator P_μ with the properties:

$$-i[P_\mu, O^{\mathrm{inc}}(x)] = \frac{\partial O_1^{\mathrm{inc}}(x)}{\partial x_\mu} \,, \tag{8a}$$

$$-i[P_\mu, O(x)] = \frac{\partial O(x)}{\partial x_\mu} \,, \tag{8b}$$

[4] Taking $f_\nu(z)$ as $\exp i[\boldsymbol{kz} - \omega_k z_0]$, a corresponding equation in momentum space is obtained.

[5] See eqs. (A.6) and (A.7). The four dimensional δ-functions express the same property in momentum space.

where $O^{\text{inc}}(x)$ and $O(x)$ represent any incoming and Heisenberg operator respectively. Therefore the invariance property (8*b*) can be explicitely incorporated in eq. (7) by taking advantage of the fact that

$$j_\nu(z) = \exp\left[iP_\mu(z_\mu - x_\mu)\right] j_\nu(x) \exp\left[-iP_\mu(z_\mu - x_\mu)\right] . \tag{9}$$

Eq. (9) explains clearly why the integration on dz_0 in eq. (7) gives rise to energy denominators containing the observed masses ([6]).

That eq. (7) does not determine completely $j_\mu(x)$ can be easily seen by splitting the expansion of $j_\mu(x)$ in normal products of $A_\nu^{\text{inc}}(x)$ as follows

$$j_\mu(x) = j_\mu^{(0)}(x) + j_\mu^{(A)}(x) , \tag{10}$$

where $j_\mu^{(0)}(x)$ is the part of $j_\mu(x)$ not containing A^{inc}. Then eq. (7) reads

$$\begin{aligned}[j_\mu^{(A)}(x), A_\nu^{\text{inc}}] = &-i\int \mathrm{d}^4z [j_\mu^{(0)}(x), j_\nu^{(0)}(z)] f_\nu(z)\eta(x-z) - \\ &-i\int \mathrm{d}^4z \left\{[j_\mu^{(0)}(x), j_\nu^{(A)}(z)] + [j_\mu^{(A)}(x), j_\nu^{(0)}(z)] + [j_\mu^{(A)}(x), j_\nu^{(A)}(z)]\right\} f_\nu(z)\eta(x-z) .\end{aligned} \tag{11}$$

If $j_\mu^{(0)}(x)$ is supposed to be known and small eq. (11) can be taken as defining uniquely an iteration procedure ([7]).

The possibility of having information about $j_\mu^{(0)}(x)$ is given by the fact that, in quantum electrodynamics $j_\mu(x)$ must satisfy the continuity equation

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0 , \tag{12}$$

and the related condition

$$[j_\mu(\boldsymbol{x}, t), j_0(\boldsymbol{x}', t)] . \tag{13}$$

In order to understand how these conditions can be used, let us start with a lowest approximation for j_μ in which $j_\mu^{(A)} = 0$ and $j_\mu^{(0)}$ is chosen such that eqs. (12), (13) are satisfied (for example $\exp[\overline{\psi}^{\text{inc}}(x) j_\mu \psi^{\text{inc}}(x)]$). By inserting this value in eq. (11) the lowest non vanishing approximation for $j_\mu^{(A)}$ is derived. The total j_μ obtained in such a way does not satisfy any more eq. (13), therefore a correction to the first choice of $j_\mu^{(0)}$ is necessary. This procedure of using eq. (7) to get the corrections to $j_\mu^{(A)}$ and eq. (13) for the corrections to $j_\mu^{(0)}$ can in principle be iterated any number of times. Whether eqs. (7), (12) and (13) are sufficient to determine completely $j_\mu(x)$ has not yet been studied in detail.

([6]) Obviously from (8*a*) it follows that the four vector P expressed as a function of incoming operators, is simply the energy-momentum four vector of the non-interacting fields with the observed masses.

([7]) The solution defined by this iteration procedure is unique, although there may well be other solutions of eq. (11) which do not tend uniformly to zero when $j_\mu^{(0)}$ tends to zero. Examples of such a situation have been found by CASTILLEJO, DALITZ and DYSON for the fixed source theory. We thank these authors for sending us a preprint of their paper.

The usefulness of eqs. (12) and (13) for the determination of $j_\mu^{(0)}$ can also be understood from the following argument. As pointed out in I, eq. (7) alone does not give any information about $j_\mu^{(0)}$ because no explicit assumption has been made about the interaction Hamiltonian b, except its linearity in the electromagnetic field. On the other hand we know that eqs. (12) and (13) impose severe restrictions on the possible form of the current j_μ as a function of $\psi(x)$. But these restrictions must also lead, through the interaction Hamiltonian and the equations of motion, to similar restrictions in the relationships between Heisenberg and incoming field operators.

An Example of the Associated Production of a Heavy Meson and a Hyperon.

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(ricevuto il 31 Agosto 1955)

Among the secondary particles of a large star, (type $22 + 36\alpha$), we have found a heavy meson which decays in the K_μ mode and a hyperon which decays in the Σ mode (1). This event occurred in a small stack of stripped emulsions which was exposed and processed in England, by the Bristol group, and is being analysed in Copenhagen, under the direction of Professor J. K. Bøggild.

The K-meson comes to rest after 51 mm. Its secondary particle is emitted with $p\beta c = (194 \pm 15)$ MeV and leaves the stack after 63 mm. The grain density of the secondary track, measured in comparison with nearby electron tracks, is (0.98 ± 0.02) times plateau at a distance of 52 mm from the decay point. The χ-decay mode is ruled out, since in that case the grain density of the secondary would be 1.2 times plateau at the distance in question. The observations are consistent with K_μ-decay.

The hyperon decays in flight after 1.06 mm. A lightly ionizing secondary particle is emitted at an angle of 28.8°, with initial $p\beta c = (227 \pm 25)$ MeV, and leaves the stack after 40 mm. Using the method of Fowler and Perkins (2), we compared the ionization of the hyperon with that of the K-meson and the α-particle primary of the star, with the result $g^* = 5.1$ times plateau and $\beta = 0.36$. The hyperon mass by ionization-scattering is $(1.9 \pm 0.8)\, m_p$. Assuming the decay scheme $\Sigma^\pm \to \pi^\pm + n$, we find $Q = (105 \pm 17)$ MeV, in good agreement with the accepted value.

The ionization-range relation for our stack is given by fig. 2 of reference (2). Since the scattering constant has not been determined for this stack, we have used values given by Voyvodic and Pickup (3), increased by 2 per cent to allow for electron scattering, and we have included a nominal value of ± 6 per cent in the scattering errors to allow for the uncertainty in the calibration.

* * *

The K-meson was found by Miss E. Trolle. Mag. scient. N. Brene has assisted with the measurements. W. C. G. Ortel was aided by a fellowship granted by the U. S. National Science Foundation.

(1) A recent account of the properties of K-mesons and hyperons may be found in *Nature*, **175**, 971 (1955).

(2) P. H. Fowler and D. H. Perkins: *Phil. Mag.*, **46**, 587 (1955).

(3) L. Voyvodic and E. Pickup: *Phys. Rev.*, **85**, 91 (1952).

Nuclear Production of Heavy Unstable Particles.

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(ricevuto il 2 Settembre 1955)

The emission of the new heavy particles from nuclei depends on the interaction of these particles with nuclear matter, as well as on the elementary production process. By using currently available experimental information on the production of unstable particles in hydrogen, some properties of the interaction of hyperons and K-Mesons with nucleons may be deduced. The present note reports on a Monte Carlo calculation constructed for this purpose in which Y (hyperon) (*) and K-particles produced by collisions of 1.9 GeV π^--mesons with Pb and C targets are followed through a succession of intranuclear collisions until they are «absorbed» (as described below) or emerge from the nucleus.

The model for our production process is taken from the observations of FOWLER *et al.* [1] on the hydrogen reaction

$$\pi^- + p \to \Lambda^0 + \theta^0 . \tag{1}$$

The subsequent interactions of the unstable particles are assumed to be

$$K + N \to K + N , \tag{2}$$

$$K + N \to Y + \pi , \tag{3}$$

$$Y + N \to Y + N , \tag{4}$$

where N denotes nucleon; the additional possibility

$$Y + N \to K + N + N , \tag{5}$$

is neglected because of its high threshold of ~ 750 MeV. The above reactions are all consistent with the associated production hypothesis [2].

The pions are traced through the nucleus using parameters determined from the experimental π^--p data [3-5],

(+) Military leave of absence from Duke University, Durham, North Carolina.

(*) For the purposes of kinematics, the Y mass is taken to be that of the Λ^0.

[1] W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE and W. L. WHITTEMORE: *Phys. Rev.*, **91**, 1287 (1953); **93**, 861 (1954); additional material in press.

[2] Reaction (3) is excluded from the scheme of GELL-MANN and PAIS: *Proc. Glasgow Conf.*, (London, 1954) which forbids double hyperon production by requiring that the K produced in (1) shall be distinct from the K absorbed in (3).

[3] EISBERG, FOWLER, LEA, SHEPHARD, SHUTT, THORNDIKE and WHITTEMORE: *Phys. Rev.*, **97**, 797 (1955) (π^--p at 1.4 GeV).

[4] COOL, MADANSKY and PICCIONI: *Phys. Rev.*, **93**, 249 (1954) (π^--p at 1.4 GeV).

[5] M. M. BLOCK and E. M. HARTH: private communication (π^--p at 1.9 GeV).

until they escape or their energies fall below 1 GeV. The cross section for unstable particle production is assumed to be constant for pion energies above 1 GeV and to vanish below that energy. The distributions in energy and angle of the unstable particles are taken from the data [1] on (1). The K is assumed to undergo elastic scattering by (2) or Y conversion by (3) in each K-N interaction, and is followed up to the point of escape from the nucleus or

ing ratio between reactions (2) and (3). In this calculation each pion interaction was taken to produce a Y-K pair; the actual yields are obtained from multiplication by the relative probability for unstable particle production in a π-N collision. By assuming this probability to be 0.03, corresponding to the cross section of ~ 1 mb reported for Λ^0-θ^0 production [1], we obtain cross sections for unstable particle emission from Pb and C. These are given in Table II for

TABLE I. – *Computed yields of unstable particles and pairs* ($\alpha = 0.5$). *Columns* (*a*) *and* (*b*) *refer, respectively, to the number produced and the number emerging. In this Table it is assumed that a* Y-K *pair was produced in every pion interaction above threshold.*

	K		Y						Y-K pairs	Y-Y pairs
			Direct (1)		Converted (3)		Total (1) + (3)			
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)		
Pb 100 pions	128	67	128	61	35	28	196	89	31	10
C 200 pions	163	138	163	127	25	21	188	148	111	14

conversion into a hyperon. The branching ratio between (2) and (3) is described by a probability α for Y conversion in a K-N collision. Details of the K-N interaction in (2) and (3) are assumed to be those of the corresponding π-N processes. The histories of the Y particles are traced by the procedure followed in an earlier calculation on Λ^0 production by 1.4 GeV pions [6].

One hundred incident pions were followed in Pb and 200 in C. Table I presents the direct results of the Monte Carlo calculation for the case of 50% conversion ($\alpha = 0.5$), i.e., a 1:1 branch-

several values of the conversion probability α.

The Y/K ratios in Pb and C are listed in Table III. In Table IV the Z dependence of the separate Y and K yields is indicated as a ratio of cross sections in Pb and C. These numbers are to be compared with the geometric cross section ratio of $(208/12)^{\frac{2}{3}} = 6.7$. We may expect that absorption of Y-particles by thermal capture in the heavy nuclei will depress the cross section ratio below its geometric value, as is the case in the calculation at 1.4 GeV (Table IV). At 1.9 GeV, however, the ratio is $\geqslant$ geometric for all α, a result which may be understood by

[6] R. JASTROW: *Phys. Rev.*, **97**, 181 (1955).

TABLE II. – *Cross sections* (mb) *for emission of unstable particles from* Pb *and* C, *based on a nuclear radius of* $1.4 \cdot 10^{-13} A^{\frac{1}{3}}$, *and a probability of* 0.03 *for* Y-K *production in* π^--p *interactions.*

E_{π^-}	1.9 GeV π^-				1.4 GeV π^- (6)	
	Pb		C		Pb	C
α	Y	K	Y	K	Y	
0	39.7	83.0	6.2	8.0	23.8	5.8
0.5	58.0	43.5	7.2	6.7	...	...
1.0	64.0	26.0	8.1	5.5	...	...

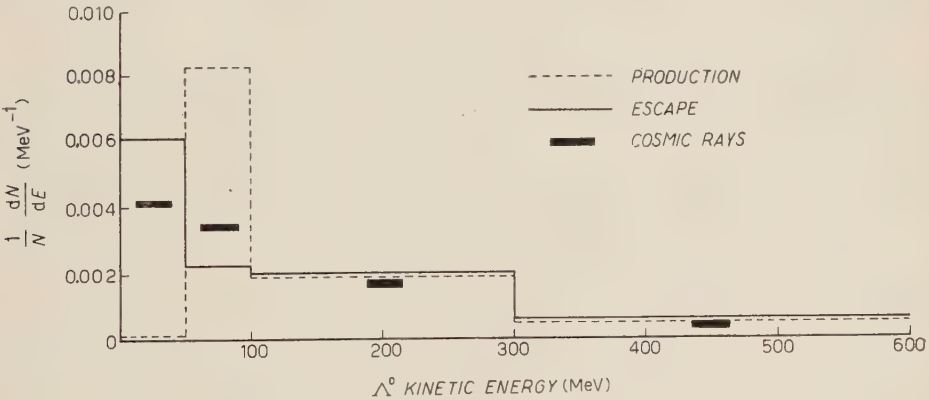

Fig. 1. – Spectrum of Y-particles produced in Pb; --- at production; — after emergence; ▬ cosmic ray observations [8-11] on Λ^0-particles.

TABLE III. – Y : K *ratio in* Pb *and* C *vs conversion probability* α.

α	0	0.5	1.00
Pb	0.50	1.33	2.50
C	0.92	1.06	1.50

noting that the procedure followed at 1.4 GeV placed the pion below the threshold for unstable particle production after one interaction, whereas in the present calculation at a higher energy several collisions, each producing a Y-K pair, are permitted the pion above threshhold. The plurality of pion interactions in the heavy nuclei then leads to a substantial increase of Y production in Pb, which compensates for the effects of absorption by thermal capture and, as indicated in Table IV, raises the ratio of Y cross sections in Pb and C to its geometric value.

Our results may be compared with the data of BLUMENFELD *et al.* [7] on π^- interactions at 1.9 GeV, which indicate: (i) a geometric ratio of cross sections for Λ^0 production in Pb and C, in agreement with Table IV, and (ii) a Λ^0- θ^0 ratio of 1.3: 1 in C and 4.5: 1 in Pb. A comparison of (ii) with Table III implies an appreciable degree of conversion ($\alpha \sim 1$). However, the number of unstable particles is small (10 Λ^0's and 7 θ^0's in C; 9 Λ^0's and 2 θ^0's in Pb), and the observed Λ^0: θ^0 ratio correspondingly uncertain; moreover, the numbers indicated in Table IV are given in terms of Y and K particles, independent of

[7] BLUMENFELD, BOOTH, LEDERMAN and CHINOWSKY: *Bull. Am. Phys. Soc.*, **30**, Z8 (1955); *Proceedings Rochester Conference* (1955).

TABLE IV. – *Ratio o fcross sections in* Pb *and* C, *for* Y *and* K *production and for several values of conversion probability* (α).

E_{π^-}	1.9 GeV			1.4 GeV [6]
α	0	0.5	1.0	0
$\left(\frac{\sigma_{Pb}}{\sigma_C}\right)$ Y	6.7	8.0	8.0	4.0
$\left(\frac{\sigma_{Pb}}{\sigma_C}\right)$ K	12.7	6.7	4.7	...

charge: if we interpret all of the identified *charged* V-particles observed in this experiment (4 in Pb, 1 in C) as K-particles and add them to the θ^0 yield, then the observed Y : K ratio is compatible with $\alpha = 0$. Thus, until more detailed experimental evidence is available, no conclusion on the magnitude of α can be reached.

Fig. 1 presents the calculated distribution of laboratory kinetic energies for Y-particles emerging from Pb, compared with a compilation of cosmic ray observations [8-11] on the energy distribution of Λ^0-particles. The calculated spectrum of Y-particles at production is also shown in Fig. 1.

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[8] BALLAM, HARRIS, HODSON, RAU, REYNOLDS, TREIMAN and VIDALE: *Phys. Rev.*, **91**, 1019 (1954).

[9] LEIGHTON, WANLASS and ANDERSON: *Phys. Rev.*, **89**, 148 (1953).

[10] FRETTER, MAY and NAKADA: *Phys. Rev.*, **89**, 168 (1953).

[11] BRIDGE, PEYROU, ROSSI and STAFFORD: *Phys. Rev.*, **91**, 362 (1953).

The Diffraction on a Conducting Wedge The General Solutions for Dipole Field.

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(ricevuto il 5 Settembre 1955)

Senior has given (1953) the method for solving the diffraction problem of a dipole field by a perfectly conducting halfplane (1). Recently I have generalized Senior's method to cover the case of a conducting wedge (2). In this paper I present the general solution of the problem for the dipole field.

The electromagnetic field of a dipole field I denote by E_k^e, H_k^e, similarly the field of a magnetic one by E_k^m, H_k^m. We have the following expressions for the diffraction field in the case of a conducting wedge with the outer angle $\varkappa$ greater than π:

$$\left.\begin{matrix} E_k^e \\ H_k^m \end{matrix}\right\} = \frac{\sin(\pi^2/\varkappa)}{4\varkappa i} \sum_{l=1,2} \left\{\begin{matrix} - \\ (-)^l \end{matrix}\right. A_s \int_{P(0)} \mathrm{d}\varrho \left[(-)^l \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_s^0} - k^2 T_{ks}^{(l)} \cdot \right.$$

$$\cdot \frac{\exp[-ikR_\varrho]}{kR_\varrho} \frac{1}{\sin(\pi/2\varkappa)(\varrho + \psi_l + \pi) \sin(\pi/2\varkappa)(\varrho + \psi_l - \pi)} +$$

$$+ \left\{\begin{matrix} - \\ + \end{matrix}\right. A_s \delta(\varphi < \pi + \varphi_0) \left[\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_s^0} - k^2 \delta_{ks} \right] \frac{\exp[-ikR]}{kR} \Big\} +$$

$$+ A_s \delta(\varphi < \pi - \varphi_0) \left[\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_s^0} - k^2 \delta_{ks}^* \right] \frac{\exp[-ikS]}{kS} +$$

$$+ A_s \delta(2\varkappa - \varphi_0 - \pi < \varphi) \left[\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_s^0} - k^2 c_{ks} \right] \frac{\exp[-ikT]}{kT} \cdot$$

(1) T. B. A. Senior: *Quart. Journ. Mech. Appl. Math.*, **4**, 101 (1953).
(2) R. Teisseyre: *Bull. Acad. Polon. Sci. Cl. III*, **3**, 157 (1955).

$$\left.\begin{matrix} H_k^e \\ E_k^m \end{matrix}\right\} = \frac{\sin(\pi^2/\varkappa)}{4\varkappa i} \sum_{l=1,2} \left\{ \begin{matrix} + \\ (-)^l \end{matrix} \right. A_s \int \mathrm{d}\varrho\, ik\varepsilon_{krp} \frac{\partial}{\partial x_r} T_{ps}^{(l)} \cdot$$

$$\cdot \frac{\exp[-ikR_\varrho]}{kR_\varrho} \frac{1}{\sin(\pi/2\varkappa)(\varrho + \psi_l + \pi) \sin(\pi/2\varkappa)(\varrho + \psi_l - \pi)} \Bigg\} -$$

$$+ A_s \delta(\varphi < \pi + \varphi_0)\, ik\varepsilon_{krs} \frac{\partial}{\partial x_r} \frac{\exp[-ikR]}{kR} -$$

$$+ \left\{ \begin{matrix} - \\ + \end{matrix} \right. A_s \delta(\varphi < \pi - \varphi_0) ik\varepsilon_{krp} \delta_{ps}^* \frac{\partial}{\partial x_r} \frac{\exp[-ikS]}{kS} \Bigg\} +$$

$$+ \left\{ \begin{matrix} - \\ + \end{matrix} \right. A_s \delta(2\varkappa - \varphi_0 - \pi < \varphi) ik\varepsilon_{krp} c_{ps} \frac{\partial}{\partial x_r} \frac{\exp[-ikT]}{kT} \Bigg\} .$$

Here: The edge of a wedge is coincident with x_3 i.e. z axis (Fig. 1). The field coordinates are $x_k = r, \varphi, z$, the dipole coordinates are $x_s^0 = r_0, \varphi_0, z_0$. The angles are $\varphi_0 + \varphi = \psi_1$, $\varphi_0 + \varphi = \psi_2$. The distances are

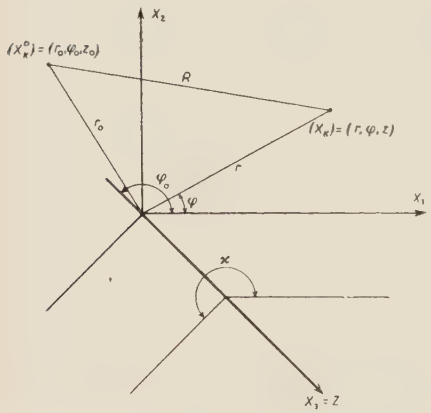

Fig. 1.

$$R_\varrho = \sqrt{r^2 + r_0^2 + 2rr_0 \cos\varrho + (z - z_0)^2}\,,$$

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos\psi_2 + (z - z_0)^2}\,,$$

$$S = \sqrt{r^2 + r_0^2 - 2rr_0 \cos\psi_1 + (z - z_0)^2}\,,$$

$$T = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\psi_1 - 2\varkappa) + (z - z_0)^2}.$$

R, S, T are the distances from the point of field x_k to the dipole, to the dipole image in the upper surface, and to the image in the lower surface respectively. The path of integration $P(0)$ is the steepest descent path through the point 0 in a complex plane. A_s is the amplitude of Hertz vector for an electric or magnetic dipole.

The time factor is always omitted. $\delta(\ldots < \ldots)$ equals unity when an inequality given in the bracket is fullfilled, and equals zero in the opposite case.

$T_{ks}^{(l)}$, c_{ks} and δ_{ks}^* have been defined as follows

$$T_{ks}^{(l)} = \begin{pmatrix} (-)^{l-1}\cos(\varrho + \psi_l) & (-)^{l-1}\sin(\varrho + \psi_l) & 0 \\ \sin(\varrho + \psi_l) & -\cos(\varrho + \psi_l) & 0 \\ 0 & 0 & (-)^l \end{pmatrix}$$

$$c_{ks} = \begin{pmatrix} \cos 2\varkappa & \sin 2\varkappa & 0 \\ \sin 2\varkappa & -\cos 2\varkappa & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\delta^{*}_{ks} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The expressions given above for E^e_k, H^e_k and E^m_k, H^m_k have an admissible singularity $(rr_0)^{-\frac{1}{2}}$ on the edge of a wedge. The discontinuities of the geometrical optic field are compensated by pure diffraction field given by the integral of the steepest descent path.

The details of this work will be given in *Acta Physica Polonica*.

A τ^- Decay with a Very Low Energy π^--Meson (*).

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(ricevuto il 9 Settembre 1955)

An interesting τ-meson decay was found in a pellicle stack exposed to particles, from a nearby target, bombarded by 6.2 GeV protons in the Berkeley accelerator. The interest in this event arises from the very low kinetic energy possessed by one of the π-mesons from the τ^--decay.

From the multiple scattering and gap density, we conclude that the τ-meson came to rest before it decayed, hence it was most probably positively charged (1). One track from the τ-decay ends after 1.45 ± 1.4 $\frac{1}{2}$m and has a low energy electron track associated with the ending. This fact together with the absence of a decay product indicates that this particle was a negative π-meson. The energy of this π^--meson is 0.53 ± 0.03 MeV as deduced from the range.

The angle between the other two lightly ionizing tracks, presumably π^+-mesons, is 172.7 ± 1.3 degrees. Since the π^--meson track is so short, the angle between it and the π^+-mesons cannot be measured accurately but it approximately bisects the angle between the two fact π-mesons. The most sensitive test for showing that this event is an ordinary τ-meson decay is to calculate the energy of the π^+-meson from the angle between the two π^+-mesons, assuming the normal Q-value of the τ^+-decay, and to compare this energy with the observed value. By this procedure we obtain an energy for the π^+-meson of $0.61^{+0.24}_{-0.20}$ MeV which is in excellent agreement with the value obtained from the range (0.53 ± 0.03 MeV).

Since the arguments ([2-4]) relating to the determination of the spin and parity

(*) Supported in part by the United States Atomic Energy Commission and by the Graduate School from founds supplied by the Wisconsin Alumni Research Foundation.

(+) On leave of absence from Brookhaven National Laboratory.

(1) In addition to the strong theoretical arguments given by Dalitz (2) regarding the improbability of the decay from rest of a negative τ-meson, experimental data is accumulating to confirm this hypothesis; for example, no τ-decays were observed from 207 K^--endings; W. F. Fry, J. Schneps, G. A. Snow and M. S. Swami: *Phys. Rev.*, in press.

(2) R. H. Dalitz: *Phys. Rev.*, **94**, 1046 (1954).

(3) R. H. Dalitz: *Phil. Mag.*, **44**, 1068 (1953).

(4) E. Fabri: *Nuovo Cimento*, **11**, 479 (1954).

of the τ^+-meson is sensitive to the low energy end of the π-spectrum, particularly that of the unlike charged π-meson, it is significant that such a low energy π^+-meson from a τ^+-decay has been found.

We are indebted to the emulsion group at the Institute for Nuclear Studies, University of Chicago for giving us the exposed plates. The cooperation and assistance given us by the many physicists at the Radiation Laboratory, University of California, is gratefully acknowledged.

The Associated Production of a χ-Meson and a Σ-Particle in a Nuclear Disintegration (*).

P. H. Fowler and D. H. Perkins

H. H. Physical Laboratory, University of Bristol

(ricevuto il 17 Settembre 1955)

During the course of a systematic analysis of the mass distribution of fast particles emerging from cosmic ray showers recorded in emulsions, an example was observed of the associated production of a Σ-particle and a χ-meson.

The parent star is of type $18+20\alpha$. The K-particle, which was originally detected by measurements of scattering and grain density along its track, comes to rest after traversing 3.61 cm. The charged secondary particle arising from its decay travels a distance of 3.9 cm through five emulsion strips before leaving. Measurements of scattering and grain density on the first 3.2 cm of track yield a mass value of 295 ± 30 m_e for the secondary, identifying it as a π-meson. The energy at emission if 108 ± 5 MeV, consistent with the accepted value for the decay scheme $\chi^+ \to \pi^+ + \pi^0$.

A charged hyperon is also produced in the parent star, and decays in flight after travelling 217 μm. The velocity of the hyperon, estimated from its ionization, is $\beta = 0.31 \pm 0.025$. The secondary particle produces a 3-prong star after travelling 3.8 cm, identifying it as a π-meson. The energy of emission in the laboratory system as calculated from ionization measurements, was 84.4 ± 2.2 MeV. The laboratory angle of emission of the secondary relative to the direction of the hyperon is $89.7° \pm 0.5°$. In the rest system of the hyperon, the emission energy of the secondary is 96.2 ± 3.3 MeV. This corresponds to a Q-value of $115 \pm \pm 5$ MeV for the 2-body decay of the hyperon $Y^\pm \to \pi^\pm + n$, and is consistent with the accepted Q-value for the Σ-particle decay.

The spatial angle at the point of emission between the Σ- and χ-particles is 92°. This angle, and the very low kinetic energy of the hyperon, cannot result from a collision of the type $\pi + N \to Y + K$, no matter what reasonable assumption be made as to the Fermi energy of the target nucleon. However, since the nuclear disintegration is produced by an α-particle, a nucleon-nucleon collision would appear more reasonable as the source of the pair of unstable particles, and is dynamically possible.

In view of the importance in either establishing some points of difference, other than decay mode, between the dif-

(*) Preliminary details of this event have been given in nature: **175**, 971 (1955).

ferent heavy mesons or else demonstrating their equality in all other respects, the information on their modes of production is of interest. To date, the Σ-particle has been observed to be produced with the τ-meson (LAL *et al.* [1]; GOTTSTEIN *et al.* [2]), the χ-meson, the $\varkappa$-meson (FRIEDLANDER *et al.* [3]) and the K_μ-meson. (Most examples of the associated production of a K-meson and a hyperon observed in emulsions probably fall within this latter class, since the K-meson secondary, although not identified, was always very lightly ionizing.) Only the K_β-particle has yet to be seen to be produced in association. The arguments of DALITZ [4] strongly suggest a difference of parity between the τ and the χ yet each has been seen in association with the Σ-particle.

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(1) D. LAL, YASH PAL and B. PETERS: *Phys. Rev.*, **92**, 438 (1953).

(2) K. GOTTSTEIN: *Nuovo Cimento*, **1**, 284 (1955).

(3) M. W. FRIEDLANDER, D. KEEFE and M. G. K. MENON: *Nuovo Cimento*, **1**, 694 (1955).

(4) R. H. DALITZ: *Procedings of the Rochester Conference* (1955).

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